Coarse Curvature by Optimal Transport
Early Stage Assessment Report

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Abstract

We offer a brief summary of existent notions of curvature in non-smooth spaces. We then focus on Ollivier's Ricci curvature and major existent results related to it. Finally, we outline possible problems to be addressed in our future research.

Contents

1 Introduction

Curvature in the setting of Riemannian geometry is an extensively studied subject and its developments have been summarized, for instance, in the books [Ber02] [Jos17].

One of the first studies of curvature in non-smooth spaces was carried out by Alexandrov towards the end of 1940's. Alexandrov's generalization to non-smooth spaces relied on the observation that triangles in spaces of positive sectional curvature have larger sum of internal angles than triangles of the same side lengths in plane. The advances in Alexandrov curvature have been summarized for example in the survey [BGP92] and the book [BBI01].

In 2003, J. Lott and C. Villani initiated their work on Ricci curvature by displacement convexity. Lott and Villani's notion of curvature was based on convexity of certain functionals of probability measures in the 2-Wasserstein space on a general metric space equipped with a measure. This was again independent of any smooth structure on the base space considered. The outcome of their research was the work [LV09]. The approach of Lott-Villani has moreover been surveyed in [Lot06] and more recently in [Vil16]. The underlying theory of optimal transport has been summarized in Villani's books [Vil03] [Vil08]. Independently, curvature by optimal transport was around the same time also studied by Sturm and von Renesse in their work [RS05].

Following the emergence of what is now referred to as Lott-Sturm-Villani theory, an alternative notion was introduced by Y. Ollivier in 2007 and published in 2009 in the article [Oll09]. In Ollivier's theory, the metric space $(\mathcal{X}, d)$ is equipped with a discrete time Markov transition kernel $m = (m_x)_{x\in\mathcal{X}}$ and curvature between any two points $x, y \in \mathcal{X}$ is defined as

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}$$

which can be construed as the negative discrete gradient in the 1-Wasserstein space along the evolution of the random walk. This definition is motivated by the general observation that two close points on a Rimennian manifold near a point of positive Ricci curvature move closer to each other when transported along parallel geodesics and drift apart near a point of negative Ricci curvature. This is then translated to the statement that two balls of uniform measure on a Riemannian manifold are closer to each other in Wasserstein distance than their centres (in the sense of geodesic distance) at points of positive Ricci curvature and further apart at points of negative Ricci curvature.

Ollivier's curvature was further studied in [OV12] [JO10] [Pau16] [JL14] and the preprint [Oll10b] offers a visual, intuitive introduction to the subject. Moreover, advances in the computability of optimal transport (e.g. [Cut13]

[PC19]) recently lead to applications of Ollivier's Ricci curvature to large scale real-world graphs [Ni+19] [Ni+15] [Ni+18] [WSJ17] [Sam+18].

Finally, we mention another recent arrival to the family of discrete curvatures, which was introduced by R. Forman in 2003 [For03] and is based on the theory of cell complexes. The properties of Forman curvature in complex networks have later been explored by [Sau+18] [Sre+16].

2 Ollivier's Ricci curvature on metric spaces

We lay out a notion of coarse curvature that is inspired by properties of parallel transport of spheres and balls on Riemannian manifolds. The coarse Ricci curvature, sometimes called Ollivier-Ricci curvature in the literature, uses for its definition larger scale properties of the space instead of the infinitesimal property of differentiation of functions near a point as in the case of common notions of curvature on Riemannian manifolds. In particular, the key insight of Ollivier [Oll09] is that curvature is related to transport distance of balls on the manifold in the limit as the centers of the two balls get arbitrarily close.

In the following, let (M, g) be a complete Riemannian manifold. Let $x \in M$ and $v, w_x \in T_xM$ and define $y := \exp_x v$ the endpoint of the geodesic starting from x in the direction of the tangent vector v. The tangent vector $w_y \in T_yM$ is one which is obtained by transporting $w_x \in T_xM$ along the geodesic between x and y to T_yM.

An observation about local behaviour of geodesics is that if the sectional curvature $K(v, w_x)$ is positive and y is close to x, the two geodesics emanating from x in the direction w_x and from y in the direction w_y will come closer to each other in the neighbourhood of x and y. Similarly, if $K(v, w_x)$ is negative then the two geodesics will drift apart locally.

Quantitatively, the sectional curvature $K(v, w)$ can be characterized by the following asymptotic relationship.

Lemma 1. [Oll09] Let $x \in M$ and $v, w_x \in T_xM$ unit tangent vectors. Let $\delta, \epsilon > 0$, $y := \exp_x \delta v$ and $w_y \in T_yM$ the unit tangent vector obtained by parallel transport of w_x from x along v to y. Then

$$d\left(\exp_x \varepsilon w_x, \exp_y \varepsilon w_y\right) = \delta\left(1 - \frac{\varepsilon^2}{2} K(v, w) + O\left(\varepsilon^3 + \varepsilon^2 \delta\right)\right)$$

as $(\delta, \epsilon) \to 0$.

The Ricci curvature $Ric(v, v)$ is the average of the sectional curvature $K(v, w)$ over w on the unit circle in the tangent space T_xM. One can then show the following analogous result for the Ricci curvature.

Lemma 2. [Oll09] Let $B_\epsilon(x), B_\epsilon(y)$ be balls of radius ϵ in T_xM and T_yM respectively. The average distance between points of $B_\epsilon(x)$ and their parallel transport images in $B_\epsilon(y)$ is

$$\delta\left(1-\frac{\varepsilon^2}{2(N+2)}\operatorname{Ric}(v,v)+O\left(\varepsilon^3+\varepsilon^2\delta\right)\right)$$

as $(\delta,\epsilon)\to 0$.

The intuition to support this result is that on Riemannian manifolds with positive curvature (that is, locally sphere-like surfaces), the transportation distance of two balls with centers that are close to each other is on average smaller than the distance of the centers.

In general metric spaces, there is no notion of either a tangent space or parallel transport. Nonetheless, Lemma 2 motivates a definition of curvature that does not rely on a smooth structure of the space. The balls $B_\epsilon(x)$ and $B_\epsilon(y)$ can be substituted with probability measures m_x, m_y with finite first moments and optimal transport distance (Wasserstein distance) W_1 can be used in place of parallel transport. This suggests the following definition of curvature for metric spaces.

Definition 3 (Ollivier's Ricci curvature)**.** [Oll09] Let $(X.d, m)$ be a metric space equipped with a Markov transition kernel $m = (m_x)_{x\in X}$. The coarse curvature between two points x and y is defined as

$$\kappa(x,y)=1-\frac{W_1(m_x,m_y)}{d(x,y)}$$

The following result from [Oll09] shows that this definition indeed coincides on Riemannian manifolds with Ricci curvature in the limit as $y\to x$ up to the factor $\frac{\epsilon^2}{2(N+2)}$ when the manifold is equipped with a random walk of step size ϵ.

Theorem 4. [Oll09] Suppose (M, g) is a complete Riemannian manifold. For an arbitrary $\epsilon > 0$, equip M with the transition kernel

$$m_x^\epsilon(dy)=\frac{\operatorname{vol}(dy)}{\operatorname{vol}(B_\epsilon(x))}$$

Then

$$\kappa(x,y)=\frac{\varepsilon^2}{2(N+2)}(\operatorname{Ric}(v,v)+O(\varepsilon)+O(d(x,y)))$$

Remark 5. Note that for ϵ and $d(x,y)$ small, the latter two terms on the right are superseded by $\operatorname{Ric}(v,v)$.

The advantage of Ollivier's curvature is that it is meaningful for strictly discrete structures like graphs. The properties of Ollivier-Ricci curvature of graphs have been studied in, for example, [OV12] [JL14] and [LLY11].

To illustrate a computation of Ollivier's Ricci curvature on a graph, consider the following example of the hypercube [Oll09].

Example 6 (Hypercube)**.** Let $X = \{0,1\}^d$ with the metric

$$d((x_1,\dots,x_d),(y_1,\dots,y_d)) = \sum_{i=1}^{d} 1_{\{x_i \neq y_i\}}$$

and suppose m is the transition kernel of a random walk that stays on the same vertex with probability $\frac{1}{2}$ and jumps to an adjacent vertex with probability $\frac{1}{2d}$. Consider two neighbouring vertices $x = (x_1,\dots,x_d)$ and $y = (y_1,\dots,y_d)$. Since they are adjacent, they differ only in one coordinate, i.e. $x_i \neq y_i$ for some i.

The following is a transport plan from m_x to m_y. The mass $\frac{1}{2} - \frac{1}{2d}$ is transported from vertex x to the adjacent vertex y. The remaining mass of $\frac{1}{2d}$ on each of the $d-1$ remaining vertices adjacent to x are moved by vector $y - x$ to neighbours of y. This yields that

$$W_1(m_x,m_y) \leqslant \left(\frac{1}{2} - \frac{1}{2d}\right) + \frac{d-1}{2d} = 1 - \frac{1}{d}$$

and hence

$$\kappa(x,y) \geqslant \frac{1}{d}.$$

Ollivier's Ricci curvature can be computed for smooth state spaces as well. However, if the underlying process is time-continuous, discretization of time is necessary to align with Definition 3.

Example 7 (Ornstein-Uhlenbeck process)**.** This is Example 9 in [Oll09]. Let $\alpha \geqslant 0, s \in \mathbb{R}$ and suppose the process $(x_t)_{t\geqslant 0}$ satisfies

$$dx_t = -\alpha x_t dt + s dW_t \in \mathbb{R}^d$$

Choosing Δt small, one can impose corresponding transition kernel on $\mathbb{R}^d$:

$$m_x(dz) = \frac{1}{(2\pi s^2)^{\frac{d}{2}}} \exp\left(-\frac{|z - x(1-\alpha\Delta t)|^2}{2s^2}\right) dz, \quad x \in \mathbb{R}$$

i.e. the normal distribution $N((1-\mathrm{e}^{-\alpha\Delta t})x, s^2 \,\mathrm{Id}_d)$ on $\mathbb{R}^d$. The variance is constant and optimal transport distance of two Gaussians with the same variance is simply the distance of the means. Hence, for any $x, y \in \mathbb{R}$,

$$\begin{aligned}\kappa(x,y) &= 1 - \frac{|(1-\mathrm{e}^{-\alpha\Delta t})x - (1-\mathrm{e}^{-\alpha\Delta t})y|}{|x-y|} \\ &= \alpha\Delta t + O(\Delta t^2)\end{aligned}$$

For curvature induced by continuous time processes as in Example 7, the following definition using continuous time Markov kernels may be more suitable, albeit more technical.

Definition 8 (Continuous time curvature)**.** [Oll09] Let (X, d, m) be a metric space equipped with a continuous time Markov process with transition kernel $m = (m_x^t)_{x \in X, t \geqslant 0}$. Define the curvature between two points $x, y \in X$ as

$$\kappa(x, y) := -\left.\frac{d}{dt}\frac{W_1(m_x^t, m_y^t)}{d(x, y)}\right|_{t=0}$$

Properties of curvature defined by a continuous time Markov process were further studied in [Vey12].

3 Existing major results

3.1 Basic tools

The following definition and lemma from [Oll09] allow extending local curvature bounds of (X, d, m) to global bounds.

Definition 9 (ϵ-geodesic spaces)**.** A metric space (X, d) is said to be ϵ-geodesic for some $\epsilon > 0$ if for all $x, y \in \mathcal{X}$ there exist $n \in \mathbb{N}$ and $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leqslant \epsilon$ for all $i = 0, \ldots, n-1$ and

$$d(x, y) = \sum_{i=0}^{n-1} d(x_i, x_{i+1})$$

Lemma 10 (Curvature in h-geodesic spaces)**.** [Oll09] Suppose $\mathcal{X}$ is an ϵ-geodesic space for some $\epsilon > 0$. If for all x, y with $d(x, y) \leqslant \epsilon$ we have $\kappa(x, y) \geqslant \kappa \in \mathbb{R}$ then κ is a lower curvature bound for *any* pair $x, y \in \mathcal{X}$.

Remark 11. Typical examples of ϵ-geodesic spaces are Riemannian manifolds and weighted graphs.

Ollivier [Oll09] lays out the following two characterizations of curvature bounded below by $\kappa > 0$.

Theorem 12 (Wasserstein distance contraction)**.** A metric space with a random walk (X, d, m) has a positive curvature bound $\kappa > 0$, i. e. $\kappa(x, y) \geqslant \kappa > 0$ for all $x, y \in X$, if and only if for all $\mu, \nu \in \mathcal{P}_1(X)$

$$W_1(\mu \star m, \nu \star m) \leqslant (1 - \kappa) W_1(\mu, \nu)$$

where $\mu \star m(dy) := \int_X m_x(dy)\mu(dx)$ and similarly for ν.

The following characterization can be proved by a simple argument involving the Kantorovich-Rubinstein duality theorem.

Theorem 13 (Lipschitz constant contraction)**.** A metric space with a random walk (X, d, m) has a positive curvature bound $\kappa > 0$ if and only if for all k-Lipschitz functions $f \in \text{Lip}(X)$ and arbitrary $k \in \mathbb{R}$ it holds that Mf is $k(1-\kappa)$-Lipschitz. In other words, X has a positive uniform curvature bound if and only if $M : \text{Lip}(X) \to \text{Lip}(X)$ is a $(1-\kappa)$-contraction in the Lipschitz norm.

The following quantity controls the diameter for spaces of positive curvature as proved by [Oll09].

Definition 14 (Jump)**.** Let (X, d, m) be a metric space equipped with a random walk. The one-step jump distance from a point $x \in X$ is defined as

$$J(x) := W_1(\delta_x, m_x) = \int_X d(x, y) m_x(dy)$$

A number of results on diameter bounds are proven under various assumptions. For example the following:

Theorem 15 (Myers theorem)**.** [Oll09] Suppose (X, d, m) is a metric space with a random walk such that $\kappa(x, y) \geqslant \kappa > 0$ for all $x, y \in X$. Then for all $x, y \in X$,

$$d(x, y) \leqslant \frac{J(x) + J(y)}{\kappa}$$

from which it can be deduced that

$$\text{diam}(X) \leqslant \frac{2 \sup_{x \in X} J(x)}{\kappa} \tag{1}$$

The preceding theorem is a metric space analogue of the classical Myers theorem from Riemannian geometry, Theorem 33 in [Ber02]:

Theorem 16 (Myers theorem)**.** Let (M, g) be a d-dimensional Riemannian manifold. Suppose there exists $\kappa > 0$ such that for all $x \in M$:

$$\inf_{v \in T_x M, |v| \leqslant 1} \text{Ric}(v, v) \geqslant \kappa$$

Then

$$\text{diam}\, M \leqslant \pi \sqrt{\frac{d-1}{\kappa}} \tag{2}$$

Remark 17. [Oll09] Note that the bound (1) loses a factor of $\frac{1}{\sqrt{\kappa}}$ in comparison to the bound (2) when considering a Riemannian manifold. It is nonetheless shown that the bound (1) is sharp in many examples. For example, a hypercube $X = \{0, 1\}^d$ has diameter d, lower curvature bound $\kappa = \frac{1}{d}$ and $J(x) = 1$ (see Example 6). Hence by Ollivier's Bonnet-Myers theorem we have $\text{diam}\, X \leqslant N$ which is precise.

3.2 Coarse Laplacian and Poincaré inequalities

Definition 18 (Averaging and Laplace operators)**.** [Oll09] Let (X, d, m) be a metric space equipped with a random walk and suppose an invariant measure ν exists. Then ν defines the space $L^2(X, \nu)/\{\text{const}\}$ (L^2 space modulo constants) with the norm

$$\|f\|^2_{L^2/\{\text{const}\}} := \operatorname{Var}_\nu f$$

Define the averaging operator $M : L^2(X, \nu)/\{\text{const}\} \to L^2(X, \nu)/\{\text{const}\}$ as

$$Mf(x) := \int_X f(y) m_x(dy)$$

and the Laplacian operator $\Delta : L^2(X, \nu)/\{\text{const}\} \to L^2(X, \nu)/\{\text{const}\}$ as

$$\Delta f(x) := Mf(x) - f(x)$$

The characterization of positive curvature in Theorem ... shows that Δ has spectral radius at most $1-\kappa$ as an operator on $\operatorname{Lip}(X)$, which is a dense subset of $L^2(X, \nu)$ for an invariant measure ν. If ν is reversible, the spectral radius bound applies to Δ acting on $L^2(X, \nu)/\{\text{const}\}$ as well:

Theorem 19 (Spectral radius and gap)**.** [Oll09] Let (X, d, m) be a metric space with a random walk and curvature bounded below by $\kappa > 0$. Suppose the invariant measure ν is reversible and the spread is finite, i. e. $\sigma < \infty$. Then for all $f \in L^2(X, \nu)/\text{const}$,

$$\operatorname{Var}_\nu Mf \leqslant (1 - \kappa)^2 \operatorname{Var}_\nu f$$

and hence the spectral radius of the operator M is at most $1 - \kappa$.

Remark 20. From the above, one can deduce that the spectral gap of $-\Delta$ is at least κ, since if λ is an eigenvalue then

$$\begin{aligned} (\operatorname{Id} - M)f &= \lambda f \\ \implies (1 - \lambda)f &= Mf \\ \implies 1 - \lambda &\leqslant 1 - \kappa \end{aligned}$$

The spectral gap is closely related to mixing properties of the random walk.

Remark 21. A related result in Riemannian geometry is the Lichnerowicz theorem for the spectral gap of the Laplacian for manifolds with a positive lower curvature bound, see Theorem 94 in [Ber02].

In a metric space setting, one may seek an interpretation of the gradient ∇f, or at least its magnitude, for $f \in L^2(X, \nu)$. Two possible definitions stem from the following corollary of the preceding theorem.

Corollary 1 (Poincaré inequalities)**.** [Oll09] Suppose (X, d, m) has curvature bounded below by $\kappa > 0$. Then

$$\operatorname{Var}_\nu f \leqslant \frac{1}{\kappa(2-\kappa)} \int_X \operatorname{Var}_{m_x} f \nu(dx) \tag{3}$$

and

$$\operatorname{Var}_\nu f \leqslant \frac{1}{2\kappa} \int_{X \times X} (f(y) - f(x))^2 \nu(dx) m_x(dy) \tag{4}$$

This suggests that a reasonable definition for the magnitude of the gradient of a function $f \in L^2(X, \nu)/\{\text{const}\}$ could be

$$|\nabla f(x)|^2 := \operatorname{Var}_{m_x} f$$

or

$$|\nabla f(x)|^2 := \frac{1}{2} \int_X (f(y) - f(x))^2 m_x(dy)$$

so that inequalities (3) and (4) were in alignment with classical Poincaré inequalities for Sobolev spaces.

3.3 Concentration and a logarithmic Sobolev inequality

The following two quantities turn out to be important for quantifying concentration of measure, as shown in [Oll09]. These results are established there with the tools of [Led01]. Again, in the following let (X, d, m) be a metric space equipped with a Markov transition kernel $m = (m_x)_{x \in X}$.

Definition 22 (Spread and local dimension)**.** Define the spread of the random walk m at $x \in X$ as

$$\sigma(x)^2 := \frac{1}{2} \int_{X \times X} d(y, z)^2 m_x(dy) m_x(dz)$$

If there is an invariant distribution ν of the random walk, define the $L^2(\nu)$ average of the spread as

$$\sigma^2 := \int_X \sigma(x)^2 \nu(dx) = \|\sigma(\cdot)\|_{L^2(X,\nu)}$$

Further define

$$\sigma_\infty(x) := \frac{1}{2} \operatorname{diam} \operatorname{supp} m_x$$

and

$$\sigma_\infty := \sup_{x \in X} \sigma_\infty(x)$$

The local dimension at $x \in X$ is defined as

$$n_x := \frac{\sigma(x)^2}{\sup\{\operatorname{Var}_{m_x} f : f \in \operatorname{Lip}_1(X)\}}$$

Theorem 23 (Concentration of measure)**.** [Oll09] Let (X, d, m) be a metric space with a random walk converging to an invariant measure ν and suppose there is a positive lower curvature bound $\kappa > 0$. If the function

$$x \mapsto \frac{\sigma(x)^2}{n_x \kappa}$$

is C-Lipschitz then there exists $t_{max} \in [0, +\infty]$ and constants $A, B > 0$ such that for all $t \leqslant t_{max}$

$$\nu(\{x \in X : f(x) + \mathbb{E}_\nu f\}) \leqslant \exp(-\frac{t^2}{A^2})$$

and for all $t \geqslant t_{max}$

$$\nu(\{x \in X : f(x) + \mathbb{E}_\nu f\}) \leqslant \exp\left(-\frac{t_{max}^2}{A^2} - \frac{t - t_{max}}{B}\right)$$

Remark 24. Ollivier [Oll09] gives an explicit form of the constants A, B and the time t_{max} as

$$A^2 = 6\mathbb{E}_\nu \left[\frac{\sigma(x)^2}{n_x \kappa}\right], \quad B = \max(2C, 3\sigma_\infty), \quad t_{max} = \frac{A^2}{3B}$$

Furthermore, it is justified in great detail by examples why the Lipschitz assumption on $\frac{\sigma(x)^2}{n_x \kappa}$ is necessary to obtain Gaussian and exponential concentration.

To formulate a log-Sobolev inequality, Ollivier [Oll09] defines the following non-local form of gradient for $f \in L^2(X, \nu)$.

Definition 25 (λ-range gradient)**.** For any $f \in L^2(X, \nu)/\{\text{const}\}$ and $\lambda > 0$ define the λ-range gradient

$$Df(x) := \sup_{y,z \in X} \frac{|f(y) - f(z)|}{d(y, z)} \exp(-\lambda(d(x, z) + d(x, y)))$$

For this definition of gradient, a Poincaré and a logarithmic Sobolev inequalies are established in [Oll09]:

Theorem 26 (Poincaré and log-Sobolev inequalities)**.** Let (X, d, m) be a metric space with a random walk and a positive lower bound on curvature $\kappa > 0$. Then for $\lambda > 0$ sufficiently small there exists $C > 0$ such that

$$\mathrm{Var}_\nu f \leqslant C \int_X (Df)^2 d\nu$$

and

$$\int_X f \log f d\nu \leqslant C \int_X \frac{(Df)^2}{f} d\nu$$

Remark 27. Ollivier gives the specific value of the constant $C = 4 \sup_{x \in X} \frac{\sigma(x)^2}{\kappa n_x}$, but also shows by example that this constant is not sharp.

3.4 The Gromov-Hausdorff topology

Ollivier [Oll09] suggests the following notion of convergence of metric spaces equipped with a random walk based on the Gromov-Hausdorff topology.

Definition 28. Let $\{(X_i, d_i, m^i)\}_{i\in\mathbb{N}}$ and (X, d, m) be compact metric spaces with random walks. We say that

$$(X_i, d_i, m^i) \to (X, d, m)$$

in the modified Gromov-Hausdorff topology if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ s. t. for every $i \geqslant N$ there are isometric embeddings $f_i : X_i \to M, f : X \to M$ to a metric space (M, d_M) such that

$$d_H(f_i(X_i), f(X)) < \epsilon$$

and moreover, for all $x \in X_i$ there is $y \in X$ such that

$$d_M(f_i(x_i), f(x)) < \epsilon \text{ and } W_1(f_i \circ m^i_x, f \circ m_y) < \epsilon$$

Here,

$$d_H(A, B) := \max\{\sup_{y\in B} \inf_{x\in A} d(x, y), \sup_{x\in A} \inf_{y\in B} d(x, y)\}$$

is the Hausdorff distance of two subsets A, B of a compact metric space (X, d), and $f_i \circ m^i_x, f \circ m_x$ are the pushforward measures from X to M by f_i and f respectively.

This definition leads to the following pointwise continuity result, which in particular implies that lower curvature bound of the form $\geqslant \kappa$ is a "closed" property.

Theorem 29. [Oll09] Suppose that $(X_i, d_i, m^i) \xrightarrow[i\to\infty]{} (X, d, m)$ in the sense of Definition 28 and let $x, y \in X$ be arbitrary. If $(x_i), (y_i)$ are sequences such that $x_i \to x$ and $y_i \to y$ as $i \to \infty$ then

$$\kappa_i(x_i, y_i) \to \kappa(x, y)$$

where κ_i is Ollivier-Ricci curvature in the space X_i.

Corollary 2. [Oll09] If there exists $\kappa \in \mathbb{R}$ such that for all $i \geqslant 1$ and all $x, y \in X_i$

$$\kappa_i(x_i, y_i) \geqslant \kappa$$

then

$$\kappa(x, y) \geqslant \kappa$$

4 Future directions

We propose some problems for further study, divided into four categories. Some of these questions may have already been addressed in the literature, in which case they contain a reference. Moreover, there is a number of important problems formulated by Ollivier in [Oll10a].

4.1 Extending results

Problem 1 (Multi-step curvature)**.** Some aspects of this problem have been studied in [Pau16]. Recall Ollivier defines curvature of a metric space with a random walk as

$$\kappa(x, y) = 1 - \frac{W_1(m_x, m_y)}{d(x, y)}$$

Let $m_x^{(k)} := m \star \ldots \star m \star \delta_x$ be the k-step transition probabilities. What can we say about the multi-step curvature

$$\kappa^{(k)}(x, y) := 1 - \frac{W_1(m_x^{(k)}, m_y^{(k)})}{d(x, y)}$$

and does there exist a limit? Clearly, by the Lipschitz contraction property, it goes to 1 if the lower bound $\kappa > 0$, but what happens in other regimes perhaps with some control on the amount of negative curvature? Consider the following settings:

- graphs (finite and infinite)
- spaces with lower bound (not necessarily positive) on curvature and some form of potential control as discussed in the previous section

What is the relationship between the evolution of curvature $\kappa^{(k)}$ as k increases and the evolution of the curvature along the discrete Ricci flow (see Problem 7)? Does one control the other?

Problem 2 (Curvature of SDEs driven by fBM)**.** Consider curvature of Euclidean spaces with respect to differential equations driven by fractional Brownian motion. Suppose that $(X_t)_{t \geqslant 0}$ is a solution to

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB_t^H \\ X_0 &= x \end{aligned}$$

where B^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1]$. Let $m_x^{\Delta t}$ be the law of $X_{\Delta t}$ for some $\Delta t > 0$ small. What is the Ollivier-Ricci curvature induced on $\mathbb{R}^d$ by the transition kernel $(m_x^{\Delta t})_{x \in \mathbb{R}^d}$?

Problem 3 (Curvature, mixing times and concentration)**.** Does Ollivier's Ricci curvature control mixing times of the random walk and vice versa? This is related to the spectral gap. In general, what is the relationship between curvature, mixing times and concentration of measure? It is known that spectral gap gives a control on mixing times. This problem was studied by Paulin in [Pau16].

Problem 4 (Global properties)**.** Can we infer global geometric or topological properties from Ollivier's Ricci curvature? One example of such a result is Ollivier's Bonnet-Myers theorem [Oll09]: if the curvature of (X, d, m) is bounded below by some $\kappa > 0$ in the sense of Ollivier, the diameter of the space is bounded (see Theorem 15 in Section 3.1). An open problem appears to be a Bishop-Gromov comparison inequality for metric spaces (Problem L in [Oll10a]). While in the Bishop-Gromov theorem one compares with a Riemannian manifold of constant Ricci curvature, Ollivier [Oll10a] suggests that hypercubes $\{0, 1\}^d$ could serve as comparison spaces of constant positive curvature.

Problem 5 (Sharpness of estimates)**.** Many estimates from the foundational work [Oll09] are metric space analogues of known results for Riemannian manifolds. A typical example of this is the Bonnet-Myers theorem as described in Remark 17. While for many discrete examples the estimate is sharp, there is a loss of a factor for Riemannian manifolds. Another example is the logarithmic Sobolev inequality of Theorem 26 which is shown in [Oll09] not to be a sharp estimate. Is it possible to sharpen the estimate, or find a condition for sharpness?

Problem 6 (Coarse sectional curvature)**.** In [Oll09], Ollivier defines the negative part of the curvature between $x, y \in X$ as

$$\kappa_-(x, y) := \frac{1}{d(x, y)} \int_{X \times X} [(d(z_1, z_2) - d(x, y)) \vee 0] \; \xi(dz_1, dz_2)$$

for an optimal coupling ξ of m_x, m_y, and further the instability between $x, y \in X$ as

$$U(x, y) := \frac{\kappa^-(x, y)}{\kappa(x, y)}$$

This quantity was of importance in establishing the logarithmic Sobolev inequality in Theorem 26, and moreover the condition $U(x, y) \geqslant 0$ was suggested in [Oll09] as a surrogate for non-negative sectional curvature in metric spaces. The condition $U(x, y) \geqslant 0$ encapsulates the situation where the random walk reduces transport distance between masses at any two points $x_1 \in \operatorname{supp} m_x$ and $y_1 \in \operatorname{supp} m_y$. This problem is mentioned in [Oll10a] as Problem P and it is suggested to compare this notion with Alexandrov's sectional curvature.

4.2 Discrete Ricci flow

Problem 7 (Discrete Ricci flow)**.** This is Problem N formulated in [Oll10a]. The Ricci flow on (X, d, m) can be defined as

$$\frac{d}{dt} d_t(x, y) = -\kappa_t(x, y) d_t(x, y)$$

This dynamics deforms the metric of the underlying space and thus also changes the curvature. Does there exist a Gromov-Hausdorff limit (or even a limit in the sense of Definition 28) and a rate of convergence? As a simpler version of this problem, one can also study the time-discretization [Ni+18] [Ni+15]

$$\text{Initialize: } d_0(x, y) := d(x, y), \quad \kappa_0(x, y) := 1 - \frac{W_1(m_x, m_y)}{d_0(x, y)}$$

For $i \geqslant 0$ do:

$$d_{(i+1)\Delta t}(x, y) := d_{i\Delta t}(x, y)(1 - \kappa_{i\Delta t}(x, y)\Delta t)$$

$$\kappa_{(i+1)\Delta t}(x, y) := 1 - \frac{W_1(m_x, m_y)}{d_{(i+1)\Delta t}(x, y)}$$

Problem 8 (Spectrum of the Laplacian)**.** Define the Laplacian [Oll09] on (X, d, m) as $M - \mathrm{Id} : L^2(\nu) \to L^2(\nu)$ where ν is an invariant measure (assuming it exists) and

$$M : L^2(\nu) \to L^2(\nu), Mf(x) := \int_X f(y) m_x(dy)$$

What happens to the spectrum of the Laplacian along the evolution of discrete Ricci flow? Again, this is more amenable to study on graphs and later on general spaces.

4.3 Curvature on graphs

Problem 9 (Random graph models)**.** What is the (average) curvature of standard generative graph models (Erdös-Rényi, Barabási-Albert, Watts-Strogatz)? For example, consider the Erdös-Rénui model $G(n, p)$. This model is a graph with n vertices and an edge between any two vertices exists with probability p. What can be said about Ollivier-Ricci curvature in terms of n and p? What is the probability of obtaining a graph with curvature at least some $\kappa \in \mathbb{R}$?

The works [JL14][LLY11][LY10] address Ollivier's curvature problems in the setting of locally finite graphs and may thus offer tools to follow through with this and other problems on graphs. Moreover, Ni et al. [Ni+15] offers an empirical study of Ollivier's curvature on these models, which shows that these models exhibit mostly negative curvature. New results for negative curvature or loosening conditions on strictly positive curvature in known results would help understanding curvature in these models [Oll10a].

Problem 10 (The idleness and decay parameters)**.** Some aspects of this problem have been studied in [LLY11]. In the context of graphs, how does the metric space change (e.g. in the Gromov-Hausdorff topology) with respect to α for the family of transition kernels

$$m_x^{\alpha}(x_i) = \begin{cases} \alpha & \text{if } x_i = x \\ (1-\alpha)/\operatorname{Deg}(x) & \text{if } x_i \sim x \\ 0 & \text{otherwise} \end{cases}$$

The parameter α is called the idleness of the random walk.

Ni et al. [Ni+19] propose to study the curvature of two-parameter random walk

$$m_x^{\alpha,p}(x_i) = \begin{cases} \alpha & \text{if } x_i = x \\ \frac{1-\alpha}{C} \cdot \exp\left(-d(x,x_i)^p\right) & \text{if } x_i \sim x \\ 0 & \text{otherwise} \end{cases}$$

What can be said about the dependency of the curvature on α and p for the space $(X, d, m^{\alpha,p})$?

4.4 Relation to other notions of curvature

Problem 11 (Relation to other notions of curvature)**.** Suppose (X, d, m) is a length space equipped with a random walk and suppose an invariant measure ν exists. The space can then be interpreted as a measured length space (X, d, ν) and analyzed using Lott-Sturm-Villani theory [LV09] [RS05]. More recently, Ambrosio, Gigli and Savaré introduced the Riemannian Ricci curvature-dimension condition $RCD(K, \infty)$ [AGS14] which has attracted much attention.

Under what conditions does the space (X, d, m) with a positive lower curvature bound satisfy curvature-dimension condition $CD(K, \infty)$ from [LV09] or even the $RCD(K, \infty)$ condition from [AGS14]? Knowing under which assumptions would allow us to connect the Ollivier-Ricci curvature with the large body of research that has been done on the curvature of measured length spaces.

Besides, there are also older notions of curvature that could be investigated with respect to Ollivier's Ricci or sectional curvatures, such as Alexandrov sectional curvature and, for graphs, Ricci curvature introduced by F. Chung and S. T. Yau in [CY96].

References

[AGS14] L. Ambrosio, N. Gigli, and G. Savaré. "Metric measure spaces with Riemannian Ricci curvature bounded from below". In: *Duke Mathematical Journal* 163.7 (2014).

[Ber02] M. Berger. *A Panoramic View of Riemannian Geometry*. Springer, 2002. ISBN: 978-3-642-18245-7.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry*. Graduate Studies in Mathematics. American Mathematical Society, 2001. ISBN: 9780821821299.

[BGP92] Y. Burago, M. Gromov, and G. Perelman. "A. D. Alexandrov spaces with curvature bounded below". In: *Russian Mathematical Surveys* 47.2 (1992).

[CY96] F. R. K. Chung and S. T. Yau. "Logarithmic Harnack inequalities". In: *Mathematical Research Letters* 3 (1996), pp. 793–812.

[Cut13] M. Cuturi. "Sinkhorn distances: lightspeed computation of optimal transportation distances". In: *NIPS'13: Proceedings of the 26th International Conference on Neural Information Processing Systems* 2 (2013), pp. 2292–2300.

[For03] R. Forman. "Bochner's Method for Cell Complexes and Combinatorial Ricci Curvature". In: *Discrete & Computational Geometry* 29 (Feb. 2003), pp. 323–374.

[Jos17] J. Jost. *Riemannian Geometry and Geometric Analyis*. 7th ed. Universitext. Springer, 2017. ISBN: 978-3-319-61859-3.

[JL14] J. Jost and S. Liu. "Ollivier's Ricci curvature, local clustering and curvature-dimension inequalities on graphs". In: *Discrete and Computational Geometry* 51.2 (2014), pp. 300–322.

[JO10] A. Joulin and Y. Ollivier. "Curvature, concentration and error estimates for Markov Chain Monte Carlo". In: *The Annals of Probability* 38.6 (2010), pp. 2418–2442.

[Led01] M. Ledoux. *The Concentration of Measure Phenomenon*. Mathematical Surveys and Monographs 89. American Mathematical Society, 2001. ISBN: 9780821828649.

[LLY11] Y. Lin, L. Lu, and S.-T. Yau. "Ricci curvature of graphs". In: *Tohoku Mathematical Journal* 63.4 (2011), pp. 605–627.

[LY10] Y. Lin and S.-T. Yau. "Ricci curvature and eigenvalue estimate on locally finite graphs". In: *Mathematical Research Letters* 17.2 (2010), pp. 343–356.

[Lot06] J. Lott. *Optimal tranport and Ricci curvature for metric-measure spaces*. Arxiv preprint: https://arxiv.org/abs/math/0610154. 2006.

[LV09] J. Lott and C. Villani. "Ricci curvature for metric-measure spaces via optimal transport". In: *Annals of Mathematics* 169.3 (2009), pp. 903–991.

[Ni+15] C. C. Ni et al. "Ricci Curvature of the Internet Topology". In: *IEEE Conference on Computer Communications (INFOCOM)* (2015), pp. 2758–2766.

[Ni+18] C. C. Ni et al. "Network Alignment by Discrete Ollivier-Ricci Flow". In: *Graph Drawing and Network Visualization, Springer International Publishing* (2018), pp. 447–462.

[Ni+19] C. C. Ni et al. "Community Detection on Networks with Ricci Flow". In: *Scientific Reports* 9.9984 (July 2019).

[Oll09] Y. Ollivier. "Ricci curvature of Markov chains on metric spaces". In: *Journal of Functional Analysis* 256.3 (Feb. 2009), pp. 810–864.

[Oll10a] Y. Ollivier. "A survey of Ricci curvature for metric spaces and Markov chains". In: *Probabilistic Approach to Geometry, Mathematical Society of Japan* (2010), pp. 343–381.

[Oll10b] Y. Ollivier. *A visual introduction to Riemannian curvatures and some discrete generalizations*. Preprint. Available from: http://www.yann-ollivier.org/rech/publs/visualcurvature.pdf. 2010.

[OV12] Y. Ollivier and C. Villani. "A Curved Brunn–Minkowski Inequality on the Discrete Hypercube, Or: What Is the Ricci Curvature of the Discrete Hypercube?" In: *SIAM Journal on Discrete Mathematics* 26.3 (2012).

[Pau16] D. Paulin. "Mixing and concentration by Ricci curvature". In: *Journal of Functional Analysis* 270.5 (2016), pp. 1623–1662.

[PC19] G. Peyré and M. Cuturi. "Computational Optimal Transport". In: *Foundations and Trends in Machine Learning* 11.5-6 (2019).

[RS05] M. K. von Renesse and K. T. Sturm. "Transport inequalities, gradient estimates, entropy and Ricci curvature". In: *Communications on pure and applied mathematics* 58.7 (2005).

[Sam+18] A. Samal et al. "Comparative analysis of two discretizations of Ricci curvature for complex networks". In: *Scientific Reports* 8.8650 (June 2018).

[Sau+18] E. Saucan et al. "Discrete Curvatures and Network Analysis". In: *Communications in Mathematical and in Computer Chemistry* 80 (2018), pp. 605–622. ISSN: 0340 - 6253.

[Sre+16] R. P. Sreejith et al. "Forman curvature for complex networks". In: *Journal of Statistical Mechanics: Theory and Experiment* 2016 (2016).

[Vey12] L. Veysseire. *Coarse Ricci curvature for continuous-time Markov processes*. 2012.

[Vil03] C. Villani. *Topics in Optimal Transportation*. Vol. 58. Graduate studies in mathematics. American Mathematical Society, 2003. ISBN: 9781470418045.

[Vil08] C. Villani. *Optimal Transport: Old and New*. Grundlehren der mathematischen Wissenschaften. Springer, 2008. ISBN: 9783662501801.

[Vil16] C. Villani. "Synthetic theory of Ricci curvature bounds". In: *Japanese Journal of Mathematics* 11 (2016).

[WSJ17] M. Weber, E. Saucan, and J. Jost. "Characterizing Complex Networks with Forman-Ricci Curvature and Associated Geometric Flows". In: *Journal of Complex Networks* 5.4 (2017), pp. 527–550.