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Whitney Extension Theorem for Regularity Structures

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Declaration

This is my own unaided work unless stated otherwise.

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Abstract

This work generalizes Whitney Extension Theorem for differentiable functions to modelled distributions in the framework of regularity structures. We lay out the principles used in the original theorem and introduce the necessary background in regularity structures theory. By drawing an analogy between C^k and $\mathcal{D}^\gamma$ spaces, we follow the steps of Whitney. Translating estimates to the context of regularity structures, we prove the existence of extensions of $\mathcal{D}^\gamma$ functions defined on closed subsets of $\mathbb{R}^d$ to $\mathcal{D}^\gamma$ functions on the entire space.

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Contents

1 Introduction

This paper generalizes Whitney's Extension Theorem [11] to an arbitrary regularity structure that contains the polynomial structure. The original theorem states that it is possible to extend a function $f : B \to \mathbb{R}$ in $\mathcal{C}^k$, $B \subset \mathbb{R}^d$ closed, to a function $g : \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{C}^k$ such that $g = f$ on B. The proof can be easily modified to apply for Hölder continuous functions, $\mathcal{C}^{k,\alpha}$, as well.

In the framework of regularity structure theory, the spaces of modelled distributions $\mathcal{D}^\gamma$ are a generalization of differentiable and Hölder continuous functions. This is made precise by the Reconstruction Theorem [6] which states that $\Pi_x f(x)$ is a local approximation of a Schwartz distribution near the point x. By drawing an analogy between Hölder spaces and $\mathcal{D}^\gamma$ spaces, we follow the steps of Whitney and prove an extension theorem for modelled distributions. Concretely, for a function $f : B \to T$ in $\mathcal{D}^\gamma$, we construct a $g : \mathbb{R}^d \to T$ in $\mathcal{D}^\gamma$ with $g = f$ on B and in the process also prove that the extension operator is continuous. In other words, for any K compact there exists an $M > 0$ such that

$$|||g|||_{\gamma,K} \leq M |||f|||_{\gamma,B\cap K_1}$$

where K_1 is the 1-fattening of K.

Regularity structures generalize the notion of differentiability of functions and their local descriptions via Taylor polynomials. With help of regularity structures, one can locally describe distributions and also define a suitable calculus of distributions, which is essential for the modern treatment of stochastic PDEs in a mathematically rigorous manner as developed by Hairer in [6][4].

1.1 Function spaces and Taylor approximations

Taylor's theorem is a classical result stating that a differentiable function defined on an open convex set $\Omega \subset \mathbb{R}^d$ can be approximated by polynomials up to precision that corresponds to the degree of differentiability of the given function.

Theorem 1.1 (Taylor's theorem)**.** [8] [11] Let Ω be open and convex. It holds for an m-times differentiable function $f : \Omega \to \mathbb{R}$ that for all $x_0 \in \Omega$,

$$f(x) = \sum_{\alpha:0\leq|\alpha|\leq m} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_0(x, x_0)$$

where the remainder R_0 satisfies the property that for all $x_0 \in \Omega, \epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B_\delta(x_0)$,

$$|R_0(x, y)| \leq \|x - y\|^m \epsilon \tag{1}$$

Moreover, if f is $(m + 1)$-times differentiable, the remainder is

$$R_0(x, x_0) = \sum_{\alpha:|\alpha|=m+1} \frac{\partial^\alpha f(x_0 + t(x - x_0))}{\alpha!}(x - x_0)^\alpha \tag{2}$$

for some $t \in (0, 1)$ dependent on x, x_0.

Remark 1.2. The property (1) implies that for all compact $K \subset \Omega$ there exists $C > 0$ such that for all $x, y \in K$,

$$|R_0(x, y)| \leq C\|x - y\|^m \tag{3}$$

This simply follows from the assumption that K is compact. K can be covered by finitely many balls given by (1) for an arbitrary fixed $\epsilon > 0$.

The following gives a concrete bound for (3).

Lemma 1.3. [8] Let Ω be an open convex set in $\mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$ be a $\mathcal{C}^{m+1}$ function for $m \in \mathbb{N}$. Then for any K compact,

$$\sup_{x,y\in K} \frac{|R_0(x,y)|}{\|x-y\|^{m+1}} \leq \frac{\sqrt{d}}{(m+1)!} \max_{\substack{z\in K\\ |\alpha|=m+1}} |\partial^\alpha f(z)|$$

Remark 1.4. Define the remainders $R_\alpha, \alpha \in \mathbb{R}^d, |\alpha| \leq n$ as the functions such that $\forall x_0 \in \Omega$,

$$\partial^\alpha f(x) = \sum_{\beta:\alpha\leq\beta,|\beta|\leq m} \frac{\partial^\beta f(x_0)}{\beta!}(x-x_0)^\beta + R_\alpha(x,x_0)$$

where all of R_α have the properties (1)(3). The property (3) can then be summed up for all of $R_\alpha, |\alpha| \leq m$ as

$$\forall K \subset \Omega \text{ compact}: \sup_{x,y\in K} \sup_{\alpha:|\alpha|\leq m} \frac{|R_\alpha|}{\|x-y\|^{m-|\alpha|}} < \infty$$

We would now like to apply Taylor's theorem to functions with a slightly more general notion of regularity.

Definition 1.5 (Locally Hölder continous functions)**.** [6] Let $\Omega \subset \mathbb{R}$ be open. For $\alpha \in [0,1)$ and $k \in \mathbb{N}$, define the space $\mathcal{C}^{k,\alpha}(\Omega)$ of k-times differentiable functions functions $f : \Omega \to \mathbb{R}$ such that for any $K \subset \Omega$ compact,

$$|f|_{\mathcal{C}^{k,\alpha},K} := \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|\partial^k f(x) - \partial^k f(y)|}{\|x-y\|^\alpha} < \infty$$

$|\cdot|_{\mathcal{C}^{k,\alpha}}$ is a semi-norm for $\mathcal{C}^{k,\alpha}(\Omega)$. $\mathcal{C}^{k,\alpha}(\Omega)$ is called the space of locally Hölder continous functions.

Definition 1.6 ($\mathcal{C}^\gamma$ spaces)**.** [6] We say $f \in \mathcal{C}^\gamma$ if f is $(\lceil\gamma\rceil - 1)$-times differentiable and for all $K \subset \mathbb{R}^d$ compact and $|\alpha| = \lceil\gamma\rceil - 1$,

$$|f|_{C^\gamma,K} := \sup_{\substack{x,y\in K\\ \|x-y\|\leq 1}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{\|x-y\|^{\gamma+1-\lceil\gamma\rceil}} < \infty \tag{4}$$

Note that for γ integer-valued we only assume that $f \in \mathcal{C}^\gamma$ is $(\gamma-1)$-times differentiable and derivatives of order $\gamma - 1$ are Lipschitz continuous.

Remark 1.7. [6] For integer values of γ, the space $\mathcal{C}^{\gamma,0}$ is strictly contained in $\mathcal{C}^\gamma$ because the latter also contains functions whose $(\gamma-1)$-th derivatives are Lipschitz continuous, but not necessarily differentiable.

Theorem 1.8 (Taylor's theorem for $\mathcal{C}^\gamma$)**.** [8] [11] Let $\Omega \subset \mathbb{R}^d$ be open and convex. It holds for a $\mathcal{C}^\gamma$ function $f : \Omega \to \mathbb{R}$ that for all $x_0 \in \Omega$,

$$f(x) = \sum_{\alpha:0\leq|\alpha|\leq\lceil\gamma\rceil-1} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + R_0(x,x_0)$$

where the remainder R_0 satisfies the property that for all compact $K \subset \Omega$ there exists $C > 0$ such that for all $x, y \in K$,

$$|R_0(x,y)| \leq C\|x-y\|^\gamma \tag{5}$$

Proof. Let $m+1 = \lceil \gamma \rceil - 1$ and use Theorem 1.1 with remainder (2) and $y := x_0 + t(x - x_0)$,

$$\begin{aligned} f(x) &= \sum_{\alpha:0\leq|\alpha|\leq m} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + \sum_{\alpha:|\alpha|=m+1} \frac{\partial^\alpha f(y)}{\alpha!}(x-x_0)^\alpha \\ &= \sum_{\alpha:0\leq|\alpha|\leq m+1} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + \sum_{\alpha:|\alpha|=m+1} \frac{\partial^\alpha f(y) - \partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha \end{aligned}$$

Since $\partial^\alpha f$ satisfies (4) for $\alpha : |\alpha| = m+1$, it holds that $\forall K$ compact $\exists C > 0$ such that $\forall x, x_0 \in K$

$$|R_0(x,x_0)| = \left| \sum_{\alpha:|\alpha|=m+1} \frac{\partial^\alpha f(y) - \partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha \right| \leq C|f|_{C^\gamma,K}\|x-x_0\|^\gamma$$

□

2 Whitney Extension Theorem

We first mention a related, well-known result in topology to illustrate the general idea.

Theorem 2.1 (Tietze Extension Theorem)**.** [12] If a topological space X is normal then for all $B \subset X$ closed and all continuous $f : B \to \mathbb{R}$ there exists a continuous extension $g : X \to \mathbb{R}$, i. e.

$$g(x) = f(x) \text{ for all } x \in B$$

Furthermore, if $\sup_{x\in B} |f(x)| \leq C$ then $\sup_{x\in X} |g(x)| \leq C$.

Whitney's Extension Theorem is a version of Theorem 2.1 for a function defined on a closed subset $B \subset \mathbb{R}^d$ which is k-times differentiable on B in a suitable sense adjusted to the fact that B is not necessarily open.

2.1 Preliminaries and statement

Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$. We present the setup laid out by Whitney in [11]. Taylor's theorem and Lemma 1.3 inspire the following generalization of differentiability.

Definition 2.2 (Differentiability on closed subsets)**.** [11] Let $B \subset \mathbb{R}^d$ be a closed set. A function $f : B \to \mathbb{R}^d$ is m-times differentiable in B if there exist functions $\{f_\alpha : |\alpha| \leq m\}$ such that for all $\alpha \in \mathbb{N}^d : |\alpha| \leq m$, $x_0 \in B$,

$$f_\alpha(x) = \sum_{\beta:0\leq|i|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x-x_0)^\beta + R_\alpha(x,x_0) \tag{6}$$

where the remainders R_α satisfy that for all $x_0 \in B, \epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B_\delta(x_0) \cap B$,

$$|R_\alpha(x,y)| \leq \|x-y\|^{m-|\alpha|}\epsilon \tag{7}$$

We will call the space of such functions $C^\alpha(B)$ and the functions f_α the derivatives of f. Define the approximations of f_α at the base point x_0 of order $m - |\alpha|$,

$$\psi_\alpha(x,x_0) := \sum_{\beta:0\leq|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x-x_0)^\beta$$

Remark 2.3. The derivatives f_α are not necessarily unique. For example, if B is a finite collection of isolated points then (6) is satisfied for any choice of functions f_α. Choosing $\delta > 0$ small enough, $B_\delta(x_0)$ consists only of x_0 so (7) trivially holds.

Remark 2.4. By Taylor's theorem, the definition of differentiability in the sense of Definition 2.2 is consistent with the classical definition. In the interior of B, it holds that for all $\alpha : |\alpha| \leq m$, $f_\alpha = \partial^\alpha f$ in $\mathring{B}$. Conversely, if $U \subset \mathbb{R}^d$ is open, $B \subset U$ is closed and $f : U \to \mathbb{R}$ is differentiable then $f_\alpha := \partial_\alpha f$ is a collection of derivatives of f on B.

Lemma 2.5 (Approximation with different base points)**.** [11] For all $x \in \mathbb{R}^d \setminus B$ and $y, z \in B$,

$$\psi_\alpha(x, y) - \psi_\alpha(x, z) = \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{R_{\alpha+\beta}(y, z)}{\beta!}(x - y)^\beta$$

Proof. [11] Consider $\psi_\alpha(x, z)$ as a function on x. Then $\partial_x^\beta \psi_\alpha(x, z) = \psi_{\alpha+\beta}(x, z)$. Since $\psi_\alpha(x, z)$ is a polynomial in x of order $m - |\alpha|$, expanding it at base point y yields

$$\psi_\alpha(x, z) = \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{\psi_{\alpha+\beta}(y, z)}{\beta!}(x - y)^\beta$$

Then

$$\begin{aligned}
\psi_\alpha(x, y) &= \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{(x - y)^\beta}{\beta!} f_{\alpha+\beta}(y) \\
&= \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{(x - y)^\beta}{\beta!} \left(\sum_{\gamma:|\gamma|\leq m-|\alpha+\beta|} \frac{f_{\alpha+\beta+\gamma}(z)}{\gamma!}(y - z)^\gamma + R_{\alpha+\beta}(y, z) \right) \\
&= \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{(x - y)^\beta}{\beta!} \left(\psi_{\alpha+\beta} + R_{\alpha+\beta}(y, z)\right) = \psi_\alpha(x, z) + \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{R_{\alpha+\beta}(y, z)}{\beta!}(x - y)^\beta
\end{aligned}$$

□

Theorem 2.6 (Whitney's Extension Theorem)**.** [11] Let $B \subset \mathbb{R}^d$ be closed and $f : B \to \mathbb{R}$ be m-times differentiable on B in the sense of Definition 2.2 with given derivatives $\{f_\alpha : \alpha \in \mathbb{N}^d, 0 \leq |\alpha| \leq m\}$. Then there exists a $g : \mathbb{R}^d \to \mathbb{R}$ such that

1. g is smooth in $\mathbb{R}^d \setminus B$
2. g is m-times differentiable in $\mathbb{R}^d$
3. $\partial^\alpha g(x) = f_\alpha(x)$ for all $\alpha : 0 \leq |\alpha| \leq m$, $x \in B$

Theorem 2.6 was proved by Whitney in [11] by a type of extrapolation via Taylor expansion terms and a particular choice for the partition of unity of $\mathbb{R}^d \setminus B$. We will loosely follow the steps in [11] in our proof of the Modelled Distribution Extension Theorem (Theorem 4.1). Hence, we present the key concepts of the proof that will be used later.

2.2 A subdivision of the complement

We will construct recursively a subdivision of $\mathbb{R}^d \setminus B$ via dyadic cubes with arbitrarily small side length near the boundary of B. [11] Define the closest distance between two sets as $\rho : 2^{\mathbb{R}^d} \times 2^{\mathbb{R}^d} \to \mathbb{R}$,

$$\rho(X, Y) := \inf_{x\in X, y\in Y} \|x - y\|$$

First, divide all of $\mathbb{R}^d$ into closed cubes of side length 1, with vertices placed on integer coordinates $\mathbb{Z}^d$. Let S_0 be the subset of these cubes such that for all cubes $C \in S_0$,

$$\rho(C, B) \geq 6 \cdot \operatorname{diam}(C) = 6\sqrt{d}$$

Clearly $\bigcup_{C \in S_0} C$ covers a subset of $\mathbb{R}^d \setminus B$. Now suppose we have defined the collections $S_0, \ldots, S_{k-1}$. Divide all of $\mathbb{R}^d$ into closed cubes of side length 2^{-k} with vertices placed on the dyadic grid $2^{-k}\mathbb{Z}^d$. Define S_k to be the subset of these cubes such that for all $C \in S_k$,

$$C \notin \bigcup_{i=1}^{k-1} S_i$$

and

$$\rho(C, B) \geq 6\sqrt{d} \cdot 2^{-k}$$

By construction, the interiors of the cubes in $S := \bigcup_{k \in \mathbb{N}} S_k$ are disjoint. Equivalently, the cubes are almost disjoint in the sense of Lebesgue measure. Since any point of $\mathbb{R}^d \setminus B$ lies in an arbitrarily small dyadic cube, we have

$$\mathbb{R}^d \setminus B = \bigcup_{C \in S} C$$

so that S subdivides all of $\mathbb{R}^d \setminus B$. An example of the presented subdivision is illustrated in Figure 1. This construction yields useful bounds for the distances of the cubes in the subdivision to the closed set B.

Proposition 2.7. [11] For all integers $k \geq 1$ and all $C \in S_k$,

$$6\sqrt{d} \cdot 2^{-k} \leq \rho(C, B) < 13\sqrt{d} \cdot 2^{-k}$$
$$6\sqrt{d} \cdot 2^{-k} \leq \max_{x \in C} \operatorname{dist}(x, B) < 14\sqrt{d} \cdot 2^{-k}$$

Proof. [11] The lower bounds are clear from the construction. Moreover,

$$\operatorname{diam}(C) = \sqrt{d} \cdot 2^{-k}$$

As $C \notin S_{k-1}$, it is contained in a dyadic cube D of side length 2^{-k+1} which does not lie in S_{k-1}. Thus

$$\rho(D, B) < 6\sqrt{d} \cdot 2^{-k+1}$$

Since $C \subset B$ and both are dyadic cubes of successive dyadic subdivisions of $\mathbb{R}^d$, we have

$$\rho(C, B) \leq \rho(D, B) + \operatorname{diam}(C) < 13\sqrt{d} \cdot 2^{-k}$$

which also implies

$$\max_{x \in C} \operatorname{dist}(x, B) \leq \rho(C, B) + \operatorname{diam}(C) < 14\sqrt{d} \cdot 2^{-k}$$

□

As an important consequence, a cube in S_k can only be adjacent to other cubes of S_k and to cubes of either S_{k-1}, or S_{k+1}, but not both.

Corollary 2.8. [11] A cube $C \in S_k$ is separated from any cube $D \in S_{k+2}$ by at least six cubes of S_{k+1}

Proof. [11] By the previous lemma,

$$\max_{x \in D} \operatorname{dist}(x, B) < 7\sqrt{d} \cdot 2^{-k-1}$$

and

$$\rho(C, B) \geq 12\sqrt{d} \cdot 2^{-k-1}$$

because $C \in S_k$. Thus we conclude that

$$\max_{x \in D} \operatorname{dist}(x, C) \geq \rho(C, B) - \max_{x \in D} \operatorname{dist}(x, B) > 5\sqrt{d} \cdot 2^{-k-1}$$

and $\sqrt{d} \cdot 2^{-k-1}$ is precisely the diameter of cubes in S_{k-1}. □

Lemma 2.9. [11] Define $\theta : \mathbb{R}^d \to \mathbb{R}$ as

$$\theta = \begin{cases} 2(1 - x_1^2) \dots (1 - x_d^2) - 1 & \text{for } x \in [-1, 1]^d \\ 0 & \text{otherwise} \end{cases}$$

Then the function $\Theta : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ defined as

$$\Theta(x) = \begin{cases} \exp\left(\frac{\theta(x)}{1-\theta(x)^2}\right) & \text{for } x \in (-1, 1)^d \setminus \{0\} \\ \infty & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is smooth in $\mathbb{R}^d \setminus \{0\}$, compactly supported in $[-1, 1]^d$, $\Theta > 0$ on $(-1, 1)^d$ and $\Theta = 0$ on $\partial[-1, 1]^d$.

Proof. Clearly Θ is smooth in $(-1, 1)^d \setminus \{0\}$. Smoothness at $\partial(-1, 1)^d$ can be proved as follows. Set $z := \frac{1}{1-\theta(x)^2}$. By induction, it can be shown that for every multiindex α there exists a polynomial P in x and z such that

$$\partial_\alpha \Theta(x) = P(x, z)\, \Theta(x) \quad \text{for } x \in (-1, 1)^d$$

If $x \to x_0 \in \partial(-1, 1)^d$ from inside of $(-1, 1)^d$, then $z \to -\infty$. Thus $|P(x, z)|$ approaches ∞ at most polynomially fast. But $\Theta(x) = \exp(xz)$ approaches 0 exponentially fast, so we can conclude that

$$\lim_{\substack{x \to x_0 \\ x \in (-1,1)^d}} \partial_\alpha \Theta(x) = 0$$

□

We are now in a position to construct a partition of unity $(\phi_\mu)_{\mu \in \mathbb{N}}$ for $\mathbb{R}^d \setminus B$ consisting of scaled and translated versions of Θ, which are further normalized in the sense that they sum up to 1 everywhere on $\mathbb{R}^d \setminus B$.

Definition 2.10 (Index of subdivision mapping)**.** Label the vertices of the cubes in the subdivision as $v^\mu, \mu \in \mathbb{N}$. Define the mapping

$$k : \mathbb{R}^d \setminus B \to \mathbb{N}$$

such that $k(x)$ is the smallest subdivision index of any cube that contains x. Suppose v^μ is a vertex of such a cube $C_\mu \in S_{k(x)}$. This means C_μ is the largest cube of the subdivision S that has v^μ as a vertex and thus the other cubes containing v^μ belong to S_k or S_{k+1}.

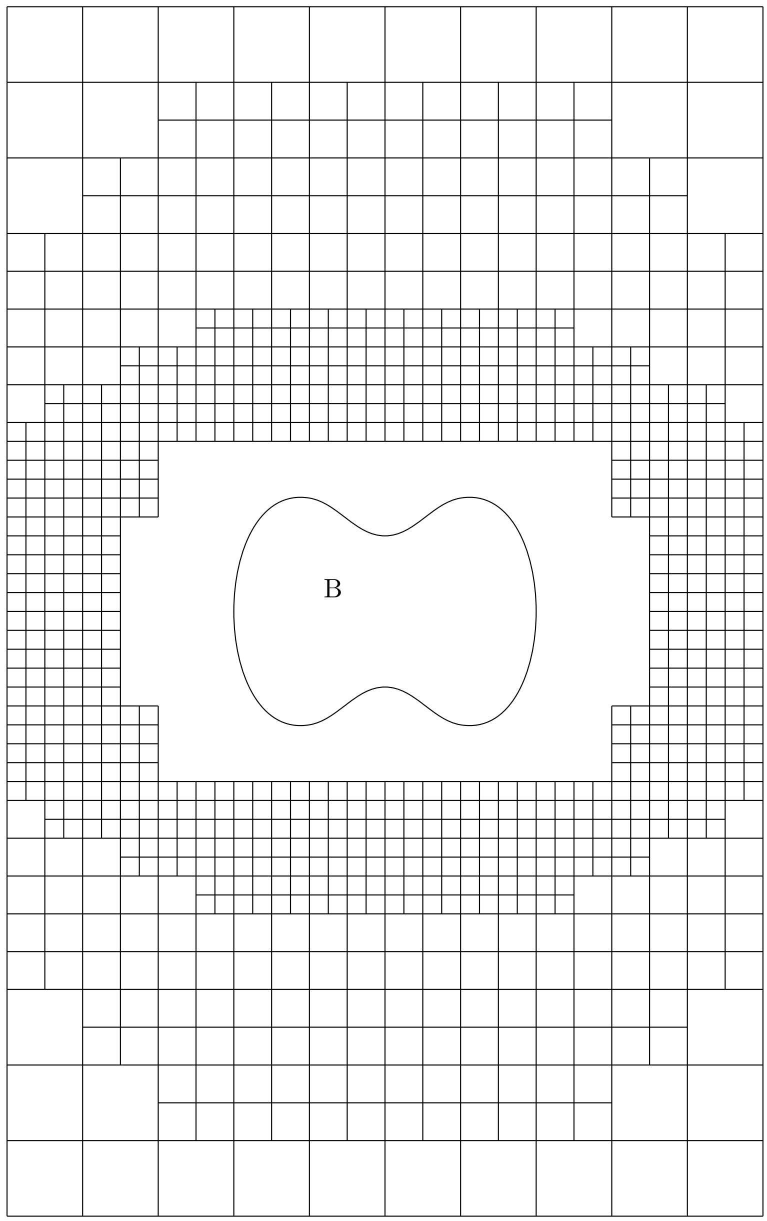

Figure 1: First three steps in the subdivision of $\mathbb{R}^d \setminus B$ yield S_0, S_1, S_2

Proposition 2.11 (A partition of unity for $\mathbb{R}^d \setminus B$)**.** [11] For any $\mu \in \mathbb{N}$ define

$$\Theta_\mu(x) := \Theta(2^{k(v^\mu)}(x - v^\mu))$$

which is clearly smooth in $\mathbb{R}^d \setminus \{0\}$ and supported in the cube $I_\mu := v^\mu + [-2^{-k(v^\mu)}, 2^{-k(v^\mu)}]^d$.

Then the functions $\phi_\mu : \mathbb{R}^d \setminus B \to \mathbb{R}$, $\mu \in \mathbb{N}$,

$$\phi_\mu(x) = \begin{cases} \frac{\Theta_\mu(x)}{\sum\limits_{\lambda \in \mathbb{N}} \Theta_\lambda(x)}, & x \neq v^\mu \\ 1, & x = v^\mu \end{cases}$$

are well-defined, smooth, compactly supported and

$$\sum_{\mu \in \mathbb{N}} \phi_\mu = 1 \text{ on } \mathbb{R}^d \setminus B$$

Proof. [11] ϕ_μ is well-defined because every point $x \in \mathbb{R}^d \setminus B$ belongs to the interior of the cube I_μ with its centre v^μ being the closest vertex to x, so $\sum\limits_{\lambda \in \mathbb{N}} \Theta_\lambda(x) > 0$. This implies $\Theta_\mu(x) > 0$ and thus $|\phi_\mu(x)| < \infty$. Since Θ_μ is supported on I_μ, so is ϕ_μ. It remains to show that ϕ_μ is smooth. For $x \neq v^\mu$, ϕ_μ is a composition of smooth functions. For $x = v^\mu$, Θ_μ is non-zero in a small neighbourhood of v^μ so

$$\phi_\mu(x) = \frac{1}{1 + \sum\limits_{\lambda : \lambda \neq \mu} \Theta_\lambda(x)}$$

in this neighbourhood, including at $x = v^\mu$ since $\Theta_\lambda(v^\mu) = 0$. This is again a composition of smooth functions. □

Note that by construction, each function ϕ_μ is compactly supported on the cube I_μ and that only finite collections of the I_μ can have a non-empty intersection.

Definition 2.12. Define the map

$$\mu : \mathbb{R}^d \setminus B \to \mathcal{P}(\mathbb{N})$$

which maps x to the finite subset of indices μ such that $x \in I_\mu$.

Remark 2.13. If x belongs to a cube in the subdivision S_k then the neighbouring cubes belong either to $S_k \cup S_{k+1}$ or to $S_k \cup S_{k-1}$. This implies that $\|x - v^\mu\| \leq 4 \cdot 2^{-k(x)}$ for all $\mu \in \mu(x)$. Moreover, there is a maximum number of cubes I_μ that contain x, i. e.

$$\forall x \in \mathbb{R}^d \setminus B : |\mu(x)| \leq C$$

There exist bounds on the derivatives of ϕ_μ depending on the degree k of the subdivision such that $C_\mu \in S_k$.

Proposition 2.14. [11] For every $j \in \mathbb{N}$ there exists $N_j > 0$ such that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = j$ and all $\mu \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}^d} |\partial^\alpha \phi_\mu(x)| < 2^{kj} N_j$$

where the support I_μ of ϕ_μ is centered at v^μ which is assumed to be a vertex of a cube in the subdivision S_k. Equivalently, using the previously defined mapping $k(\cdot)$, there exists $N_j > 0$ such that for all $x \in \mathbb{R}^d \setminus B, \mu \in \mu(x), \alpha : |\alpha| = j$,

$$|\partial^\alpha \phi_\mu(x)| < 2^{k(x)j} N_j$$

Proof. [11] Noting that $\Phi_\mu(x) = \Phi(v^\mu + 2^{-k}(x - v^\mu))$ is just a shifted and scaled version of Φ, differentiate using the chain rule to obtain

$$\partial^\alpha \Phi_\mu(x) = 2^{kj} \partial^\alpha \Phi(v^\mu + 2^{-k}(x - v^\mu))$$

and setting $N_j := \sup\limits_{\alpha : |\alpha| = j} \sup\limits_{x \in \mathbb{R}^d} |\partial^\alpha \Phi(x)| < \infty$ gives the result. □

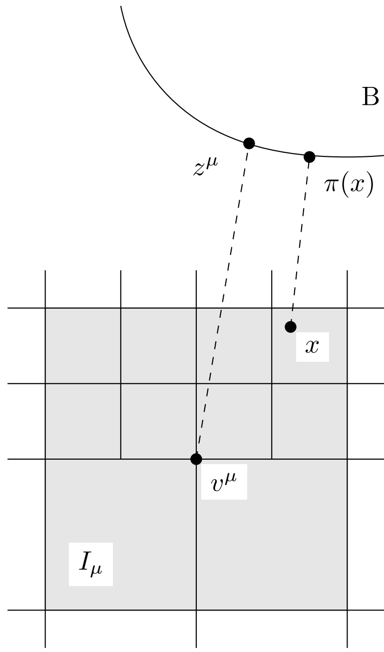

Figure 2: Labelling points in the subdivision of $\mathbb{R}^d$ and their projections on B. The shaded area I_μ is the support of ϕ_μ.

2.3 Distance estimates for the subdivision

We deduce some estimates used throughout the proofs of the original theorem and Theorem 4.1.

Definition 2.15 (Projection mapping)**.** Define the mapping $\pi : \mathbb{R}^d \setminus B \to \partial B$ that projects a point outside of B to its boundary so that

$$\|\pi(y) - y\| = \text{dist}(y, B)$$

Let $z^\mu = \pi(v^\mu) \in \partial B$ so that $\|z^\mu - v^\mu\| = \text{dist}(v^\mu, B)$.

By the triangle inequality and definition of π, it holds for all $y \in \mathbb{R}^d \setminus B, x_0 \in \partial B$,

$$\|\pi(y) - x_0\| \leq \|y - \pi(y)\| + \|y - x_0\| \leq 2\|y - x_0\| \tag{8}$$

Moreover, π maps bounded sets to bounded sets: for all y in a bounded set K,

$$\|\pi(y)\| \leq \|\pi(y) - y\| + \|y\| \leq \|y - x_0\| + \|y\| \leq \|x_0\| + 2 \sup_{y \in K} \|y\|$$

Proposition 2.16 (Distance estimates)**.** [11] There exist $C_1, C_2 > 0$ such that $\forall y \in \mathbb{R}^d \setminus B, \mu \in \mu(y)$,

$$C_1 \cdot 2^{-k(y)} \leq \|y - \pi(y)\| \leq C_2 \cdot 2^{-k(y)}$$
$$C_1 \cdot 2^{-k(y)} \leq \|v^\mu - z^\mu\| \leq C_2 \cdot 2^{-k(y)}$$
$$C_1 \cdot 2^{-k(y)} \leq \|y - z^\mu\| \leq C_2 \cdot 2^{-k(y)}$$
$$C_1 \cdot 2^{-k(y)} \leq \|v^\mu - \pi(y)\| \leq C_2 \cdot 2^{-k(y)}$$

Proof. It follows from Proposition 2.7 that

$$6\sqrt{d} \cdot 2^{-k(y)} \leq \|y - \pi(y)\| < 14\sqrt{d} \cdot 2^{-k(y)}$$

For all $\mu \in \mu(y)$ we have either $k(v^\mu) = k(y)$ or $k(v^\mu) = k(y) - 1$. In both cases it follows that

$$6\sqrt{d} \cdot 2^{-k(y)} \leq 6\sqrt{d} \cdot 2^{-k(v^\mu)} \leq \|v^\mu - z^\mu\| \leq 14\sqrt{d} \cdot 2^{-k(v^\mu)} \leq 28\sqrt{d} \cdot 2^{-k(y)}$$

By Remark 2.13, $\|y - v^\mu\| \leq 4 \cdot 2^{-k(y)}$ and thus

$$6\sqrt{d} \cdot 2^{-k(y)} \leq \|y - \pi(y)\| \leq \|y - z^\mu\| \leq \|y - v^\mu\| + \|v^\mu - z^\mu\| \leq 32\sqrt{d} \cdot 2^{-k(y)}$$

and similarly

$$6\sqrt{d} \cdot 2^{-k(y)} \leq \|v^\mu - z^\mu\| \leq \|v^\mu - \pi(y)\| \leq \|y - v^\mu\| + \|y - \pi(y)\| \leq 18\sqrt{d} \cdot 2^{-k(y)}$$

□

Corollary 2.17 (More estimates)**.** There exists $C > 0$ such that $\forall y \in \mathbb{R}^d \setminus B, \mu \in \mu(y), x_0 \in B$,

$$\begin{aligned} \|v^\mu - z^\mu\| &\leq C\|y - x_0\| \\ \|y - z^\mu\| &\leq C\|y - x_0\| \\ \|v^\mu - \pi(y)\| &\leq C\|y - x_0\| \\ \|z^\mu - \pi(y)\| &\leq C\|y - x_0\| \end{aligned}$$

Proof. These follow easily from the previous proposition. For example,

$$\|v^\mu - z^\mu\| \leq C_2 \cdot 2^{-k(y)} \leq \frac{C_2}{C_1}\|y - \pi(y)\| \leq C\|y - x_0\|$$

□

The following lemma gives a differentiability criterion at a point of a closed subset of an interval. This will be used to show the existence of partial derivatives of a proposed extension at the boundary points of B.

Lemma 2.18. [11] Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continous function. Suppose B^* is a closed subset of I and f is differentiable on $I \setminus B^*$. Further assume $z_0 \in I$ and $z_0' \in \mathbb{R}$ with the property that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

1. if $z \in B^*$ and $|z - z_0| < \delta$ then $\left|\frac{f(z)-f(z_0)}{z-z_0} - z_0'\right| < \epsilon$
2. if $z \in I \setminus B^*$ and $|z - z_0| < \delta$ then $|f'(z) - z_0'| < \epsilon$

Then f is differentiable at z_0 and $f'(z_0) = z_0'$

Proof. Suppose that $z_0 \in I \setminus B$ so that $f'(z_0)$ exists. Choosing $z = z_0$, the second property gives that $\forall \epsilon > 0 : |f'(z_0) - z_0'| < \epsilon$, thus $f'(z_0) = z_0'$.
Suppose $z_0 \in B$ and it is an isolated point of B. This means there exists $\delta_0 > 0$ such that $z \in I \setminus B$ for all $z \in I$ with $0 < |z - z_0| < \delta_0$. We show that f is right differentiable at z_0 and left differentiability will follow similarly. Let (h_k) be an arbitrary sequence in $\mathbb{R}$ with $0 \leq h_k < \delta_0$ and $h_k \searrow 0$. By definition of δ_0, f is differentiable on $(z_0, z_0 + h_k) \subset I \setminus B$ for all $k \in \mathbb{N}$. Hence, by the Mean Value Theorem there are $z_k \in (z_0, z_0 + h_k)$ such that

$$\frac{f(z_0 + h_k) - f(z_0)}{h_k} = f'(z_k)$$

As $h_k \searrow 0$, we have $z_k \to z_0$. Therefore, the second property implies that $f'(z_k) \to z_0'$, hence

$$\lim_{k\to\infty} \frac{f(z_0 + h_k) - f(z_0)}{h_k} = f'(z_0) = z_0'$$

Suppose $z_0 \in B$ and assume without loss of generality that B has no isolated points, because f is differentiable at isolated points as discussed in the previous case. If z_0 is a point in the interior of B then there exists $\delta_0 > 0$ such that $z \in B$ for all $z \in I$ with $|z - z_0| < \delta_0$. Then property 1 means precisely that f is differentiable at z_0 and $f'(z_0) = z_0'$. Now assume $z_0 \in \partial B$. Since B has no isolated points, there exists $\delta_0 > 0$ such that (without loss of generality)

$$\begin{cases} z \in I \setminus B & \text{for } z_0 < z < z_0 + \delta_0 \\ z \in B & \text{for } z_0 - \delta_0 < z \leq z_0 \end{cases}$$

Applying the Mean Value Theorem argument as before and using property 2, we obtain that f is right differentiable at z_0 with right derivative equal to z_0'. By property 1, f is left differentiable at z_0 with left derivative equal to z_0'. Hence the two semi-derivatives agree and $f'(z_0) = z_0'$ exists. □

The setup of sections 2.1-2.3 will also be essential in proving the Modelled Distribution Extension Theorem. We now present the proof of the original theorem discovered by Whitney in [11]. An alternative proof can be found in Hörmander's textbook [8].

2.4 Proof of Theorem 2.6

Proof. [11] Define $g : \mathbb{R}^d \to \mathbb{R}$ as

$$g(x) = \begin{cases} f(x) & \text{for } x \in B \\ \sum_{\mu \in \mathbb{N}} \phi_\mu(x)\psi(x, z^\mu) & \text{for } x \in \mathbb{R}^d \setminus B \end{cases}$$

where $\psi := \psi_0$. Note that for every point $x \in \mathbb{R}^d \setminus B$, the expression for g is a sum of finitely many terms $\mu \in \mathbb{N}$.

f is α times differentiable in $\mathring{B}$ and g is smooth in $\mathbb{R}^d \setminus B$ as a finite sum of smooth functions. Hence it remains to show g is m times continuously differentiable near ∂B. The approach is to prove that if an α-th derivative of g exists in $\mathbb{R}^d$ then it is continuous near ∂B and therefore everywhere in $\mathbb{R}^d$. Then we show that if $\partial^\alpha g$ exists for all $\alpha : |\alpha| = j < m$ then the derivative exists for all $\alpha : |\alpha| = j + 1$. The fact that $g \in \mathcal{C}^m$ will then follow by induction.

Let $x_0 \in \partial B$ and choose any $\epsilon : 0 < \epsilon < 1$. Verify that there exists $\delta > 0$ such that $\forall x \in B_\delta(x_0) \setminus B$,

$$|\partial^\alpha g(x) - f_\alpha(x_0)| < \epsilon$$

Let

$$M := \sup_{z \in B_1(x_0) \cap B} \sup_{i : |i| \leq m} |\partial^i f(z)| < \infty$$

Without loss of generality, we look for $\delta < 1$ small enough so that $\forall x, y \in B_\delta(x_0)$:

$$|f_\alpha(y) - \psi_\alpha(y, x)| = |R_\alpha(x, y)| < \epsilon \tag{9}$$

Step 1: The Taylor expansion of f_α of order $m - |\alpha|$ at $\pi(x)$ with base point x_0 is

$$\psi_\alpha(\pi(x), x_0) = \sum_{i : 0 \leq |i| \leq m - |\alpha|} f_{\alpha + i}(x_0)(\pi(x) - x_0)^i \tag{10}$$

which is a sum of at most $(m+1)^d$ terms of order $O(\|\pi(x) - x_0\|^m)$. If $\|x - x_0\| < \frac{\delta}{2}$ then $\|\pi(x) - x_0\| < \delta < 1$. Thus (10) can be estimated by

$$|\psi_\alpha(\pi(x), x_0) - f_\alpha(x_0)| \leq \sum_{i : 0 \leq |i| \leq m - |\alpha|} M \|\pi(x) - x_0)\|^i \leq (m+1)^d M \delta$$

If $\delta < \frac{\epsilon}{M(m+1)^d}$ then $\forall x \in B_\delta(x_0)$,

$$|\psi_\alpha(\pi(x), x_0) - f_\alpha(x_0)| < \epsilon$$

and from (9) we have

$$|f_\alpha(\pi(x)) - \psi_\alpha(\pi(x), x_0)| < \epsilon$$

and the triangle inequality yields

$$|f_\alpha(\pi(x)) - f_\alpha(x_0)| < 2\epsilon \tag{11}$$

Step 2: The Taylor expansion of f_α of order $m - |\alpha|$ at x with base point $\pi(x)$ is

$$\psi_\alpha(x, \pi(x)) = \sum_{i:0\leq|i|\leq m-|\alpha|} f_{\alpha+i}(\pi(x))(x - \pi(x))^i$$

If $\|x - x_0\| < \frac{\delta}{2}$ then $\|x - \pi(x)\| < \delta < 1$ and similarly as in the previous step,

$$|\psi_\alpha(x, \pi(x)) - f_\alpha(\pi(x))| < \epsilon \tag{12}$$

Step 3: Estimates (11)(12) give

$$|\psi_\alpha(x, \pi(x)) - f_\alpha(x_0)| < 3\epsilon \tag{13}$$

Step 4: We prove that there exists $C > 0$ such that for all $x : \|x - x_0\| \leq \frac{\delta}{2}$,

$$|\partial^\alpha g(x) - \psi_\alpha(x, \pi(x))| \leq C\epsilon \tag{14}$$

and then triangle inequality will give continuity of g at x_0.

Recall $\mu(x)$ is the set of indices such that $x \in I_\mu$ for all $\mu \in \mu(x)$. If $\|x-x_0\| < \frac{\delta}{2}$ then $\|z^\mu - x_0\| < \delta$ for all $\mu \in \mu(x)$ where $z^\mu := \pi(v^\mu)$ and also $\|\pi(x) - x_0\| < \frac{\delta}{2}$. Then the defining property of R_α implies

$$|R_\alpha(z^\mu, \pi(x))| \leq \|z^\mu - \pi(x)\|^{m-|\alpha|}\epsilon \tag{15}$$

Define for $\mu \in \mu(x)$,

$$\zeta_{\mu,\alpha}(x) := \psi_\alpha(x, z^\mu) - \psi_\alpha(x, \pi(x))$$

By Lemma 2.5 and applying (15) for $R_{\alpha+\beta}$,

$$\begin{aligned}
|\zeta_{\mu,\alpha}(x)| &\leq \sum_{\beta:|\beta|\leq m-|\alpha|} \left| \frac{R_{\alpha+\beta}(z^\mu, \pi(x))}{\beta!}(z^\mu - \pi(x))^\beta \right| \\
&\leq \sum_{\beta:|\beta|\leq m-|\alpha|} \frac{1}{\beta!}\|z^\mu - \pi(x)\|^{m-|\alpha|} \\
&\leq (m+1)^d \delta^{m-|\alpha|}\epsilon
\end{aligned} \tag{16}$$

Then, because (ϕ_μ) is a partition of unity for $\mathbb{R}^d \setminus B$,

$$g(x) = \sum_{\mu\in\mu(x)} \phi_\mu(x)\psi(x, z^\mu) = \psi(x, \pi(x)) + \sum_{\mu\in\mu(x)} \phi_\mu(x)\zeta_{\mu,0}(x)$$

Differentiating ψ with respect to the first argument we have $\partial^\alpha\psi(x, \pi(x)) = \psi_\alpha(x, \pi(x))$ and $\partial^\alpha\zeta_{\mu,0}(x) = \zeta_{\mu,\alpha}(x)$. Then

$$\partial^\alpha g(x) = \psi_\alpha(x, \pi(x)) + \sum_{\mu\in\mu(x)} \sum_{\beta:\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta\phi_\mu(x)\zeta_{\mu,\alpha-\beta}(x)$$

The sum on the right can be further estimated. The assumption $|\alpha| \leq m$ implies $\binom{\alpha}{\beta} \leq (m!)^d$, Proposition 2.14 gives $|\partial^\beta \psi_\mu(x)| \leq N_\alpha 2^{-k(x)|\alpha|}$ and (16) bounds $|\zeta_{\mu,\alpha-\beta}(x)|$. Hence

$$|\partial^\alpha g(x) - \psi_\alpha(x, \pi(x))| \leq N_\alpha((m+1)!)^d \cdot 2^{-k(x)|\alpha|}\delta^{m-|\alpha|-|\beta|}\epsilon \tag{17}$$

We know that by the construction of the subdivision,

$$6\sqrt{d} \cdot 2^{-k(x)} \leq \|x - \pi(x)\| \leq \|x - x_0\| \tag{18}$$

Estimates (17)(18) then yield (14).
Step 5: Estimates (13)(14) imply there exists $C' > 0$ such that $\forall x \in B_{\frac{\delta}{2}}(x_0)$,

$$|\partial^\alpha g(x) - f_\alpha(x_0)| < C'\epsilon$$

Step 6: Apply Lemma 2.18 to show differentiability of $\partial^\alpha g$ for any $\alpha : |\alpha| = j \leq m-1$. Let $x_0 \in \partial B$ and let $B^* := \{t \in \mathbb{R} : x_0 + te_i \in B\}$ which is a closed subset of $\mathbb{R}$ with $0 \in B^*$. Define $w : \mathbb{R} \to \mathbb{R}$, $w(t) = \partial^\alpha g(x_0 + te_i)$ for an arbitrary $i \in \{1, \ldots, d\}$. We verify the assumptions for Lemma 2.18 hold for the point $0 \in B^*$.

1. Note that $w : \mathbb{R} \to \mathbb{R}$ is differentiable on B^* with k-th derivatives $w_k(t) := \partial^{\alpha+ke_i} f(x_0 + te_i)$ according to Definition 2.2. If $t \in B^*$ then

$$w(t) = w(0) + \sum_{k=1}^{m-|\alpha|} \frac{w_k(0)}{k!} t^k + R_\alpha(te_i, 0)$$

which implies

$$\left| \frac{w(t) - w(0)}{t} - \partial^{\alpha+e_i} f(x_0) \right| \leq \sum_{k=2}^{m-|\alpha|} \frac{|w_k(0)|}{k!} |t|^{i-1} + \left| \frac{R_\alpha(te_i, 0)}{t} \right|$$

The first term on the right clearly converges to 0 as $t \searrow 0$ and for the second term we have that $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall t : |t| < \delta < 1$,

$$\left| \frac{R_\alpha(te_i, 0)}{t} \right| \leq |t|^{m-|\alpha|-1}\epsilon \leq \epsilon$$

since $m - |\alpha| \geq 1$. Hence $\forall \epsilon \; \exists \delta > 0$ such that $\forall t \in B^* : |t| < \delta$

$$\left| \frac{w(t) - w(0)}{t} - \partial^{\alpha+e_i} f(x_0) \right| < \epsilon$$

2. If $t \notin B^*$ then $w'(t) = \partial^{\alpha+e_i} g(x_0 + te_i)$ and from Step 6 we know that $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall t \in \mathbb{R} \setminus B^* : |t| < \delta$,

$$|w'(t) - f_{\alpha+e_i}(x_0)| = |\partial^{\alpha+e_i} g(x_0 + te_i) - f_{\alpha+e_i}(x_0)| < \epsilon$$

□

Remark 2.19. Whitney's publication [11] goes further and proves that there exist smooth extensions to smooth functions defined on closed subset. Using additional approximation results, Whitney then proves that in fact one can find extensions that are analytic in $\mathbb{R}^d \setminus B$.

3 Regularity structures

We summarize basic regularity structure theory (by Hairer [4][5][6]) in order to formulate the Modelled Distribution Extension Theorem.

3.1 Basic definitions and elementary results

Definition 3.1 (Regularity structure)**.** [6] A regularity structure is a triple $\mathcal{T} = (A, T, G)$ where:

1. A is a locally finite subset of the real line bounded below and containing 0
2. $T = \bigoplus_{\alpha \in A} T_\alpha$ is a graded vector space with T_α Banach spaces over $\mathbb{R}$ and $T_0 \approx \mathbb{R}$
3. G is a group of linear operators $\Gamma : T \to T$ such that for all $\Gamma \in G, \alpha \in A, \tau \in T_\alpha$, it holds that

$$(\Gamma - \mathrm{Id})(\tau) \in \bigoplus_{\beta < \alpha} T_\beta$$

   and for all $\Gamma \in G$,

$$\Gamma 1 = 1$$

where 1 denotes simply the element of $T_0 \approx \mathbb{R}$. We will call A the set of homogeneities, T the model space and G the structure group. For any $\gamma \in \mathbb{R}$, denote $T_\gamma^- := \bigoplus_{\alpha < \gamma} T_\alpha$.

Remark 3.2. Property 3 says that $\Gamma \in G$ maps an element of T to components of T of the same or lower order homogeneity, but not higher. A being locally finite implies that there are finitely many $\beta \in A, \beta < \gamma$. As A is locally finite and bounded below, we can define $\bar{\alpha} := \min A$.

Definition 3.3 (Projection operator)**.** [11] For any $\alpha \geq \min A$, define $\mathcal{Q}_\alpha : T \hookrightarrow T_\alpha$, the projection to the component of degree α.
For any $\gamma > \min A$, define $\bar{Q}_\gamma : T \hookrightarrow \bigoplus_{\alpha < \gamma} T_\alpha$, the projection to components of degree less than γ.

Definition 3.4 (Model for a regularity structure)**.** [6] Let $\mathcal{T} = (A, T, G)$ be a regularity structure. A model for $\mathcal{T}$ is a pair (Π, Γ) with the following specifications.

1. $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$ has the algebraic properties

$$\begin{aligned} &\forall x \in \mathbb{R}^d : \Gamma_{xx} = \mathrm{Id} \\ &\forall x, y, z \in \mathbb{R}^d : \Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \end{aligned} \tag{19}$$

   and $\forall \gamma > 0, K \subset \mathbb{R}^d$ compact $\exists C_{\gamma,K}$ such that $\forall l, m \in A, l < \gamma, m < l, a \in T_l, x, y \in K$:

$$\|\Gamma_{xy} a\|_m \leq C_{\gamma,K} \|a\| \|x - y\|^{l-m} \tag{20}$$

2. $\Pi = (\Pi_x)_{x \in \mathbb{R}^d}$ is a collection of continuous linear maps $\Pi_x : T \to \mathcal{S}'(\mathbb{R}^d)$ having the algebraic property

$$\forall x, y \in \mathbb{R}^d : \Pi_y = \Pi_x \circ \Gamma_{xy} \tag{21}$$

   and $\forall \gamma > 0, K \subset \mathbb{R}^d$ compact $\exists C_{\gamma,K}$ such that $\forall \delta \in (0, 1], l \in A, l < \gamma, a \in T_l, x \in K$:

$$|(\Pi_x a)(\delta^{-d} \varphi\left(\delta^{-1}(\cdot - x)\right))| \leq C_{\gamma,K} \|a\| \delta^l \tag{22}$$

   holds for all test functions $\varphi \in \mathcal{S}(B_1(0))$ such that $\|\varphi\|_{\mathcal{C}^r} \leq 1$ where $r = -\lfloor \min\{0, \min A\} \rfloor$.

The following lemma is a simple consequence of (20) and the fact that Γ_{xy} maps to elements of the same or lower degree of homogeneity.

Lemma 3.5. Let $\gamma > 0$ and $K \subset \mathbb{R}^d$ compact. There exists $C_{\gamma,K} > 0$ such that for all $\beta \in A$, $\beta < \gamma$, $a \in T$, $x, y \in K$:

$$\|\Gamma_{xy} a\|_\beta \leq C_{\gamma,K} \sum_{\alpha:\beta\leq\alpha<\gamma} \|a\|_\alpha \|x-y\|^{\alpha-\beta}$$

Proof. Write $a = \sum_{\alpha\in A} a_\alpha$ so that $\|a\|_\alpha = \|a_\alpha\|$. By Remark 3.2,

$$\Gamma_{xy} a = \sum_{\alpha:\beta\leq\alpha<\gamma} \Gamma_{xy} a_\alpha$$

and by triangle inequality and (20),

$$\|\Gamma_{xy} a\|_\beta \leq \sum_{\alpha:\beta\leq\alpha<\gamma} \|\Gamma_{xy} a_\alpha\|_\beta \leq C_{\gamma,K} \sum_{\alpha:\beta\leq\alpha<\gamma} \|a\|_\alpha \|x-y\|^{\alpha-\beta}$$

□

Definition 3.6 ($\mathcal{D}^\gamma$ spaces)**.** [6] We say that a function $f : \mathbb{R}^d \to T$ is in $\mathcal{D}^\gamma$ if for all $K \subset \mathbb{R}^d$ compact,

$$|||f|||_{\gamma,K} := \sup_{x\in K} \sup_{\beta<\gamma} \|f(x)\|_\beta + \sup_{\substack{x,y\in K\\ \|x-y\|\leq 1}} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|^{\gamma-\beta}} < \infty$$

$\mathcal{D}^\gamma$ is called the space of modelled distributions. It can be shown that $|||\cdot|||_{\gamma,K}$ defines a semi-norm on the equivalence classes of $\mathcal{D}^\gamma$ with the relation $f \sim g \iff \forall \alpha < \gamma : \ \mathcal{Q}_\alpha f = \mathcal{Q}_\alpha g$.

Remark 3.7. The choice of constraint $\|x - y\| \leq 1$ in the second term supremum is to some extent arbitrary. Using the constraint $\|x-y\| \leq r$ for $r > 0$ arbitrary or using no constraint gives equivalent seminorms. For the case of no constraint, this can be seen from the following.

$$\sup_{x,y\in K} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|^{\gamma-\beta}} \leq \sup_{\substack{x,y\in K\\ \|x-y\|\leq 1}} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|^{\gamma-\beta}} + \sup_{\substack{x,y\in K\\ \|x-y\|>1}} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|^{\gamma-\beta}}$$

By Lemma 3.5, definition of $|||\cdot|||_{\gamma,K}$ and the constraint $\|x - y\| > 1$, there exists $C > 0$ such that

$$\sup_{\substack{x,y\in K\\ \|x-y\|>1}} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|^{\gamma-\beta}} \leq |||f|||_{\gamma,K} + \sup_{x,y\in K} \sup_{\beta<\gamma} \sum_{\alpha:\beta\leq\alpha<\gamma} \|f(y)\|_\alpha \|x-y\|^{\alpha-\beta} \leq C|||f|||_{\gamma,K}$$

We give an essential example of a regularity structure and model which defines abstract polynomials.

Example 3.8 (The polynomial structure)**.** [6] For any $d \in \mathbb{N}$ define $\mathcal{T}_d := (\mathbb{N}, T, G)$ as follows. Let the model space be

$$T := \bigoplus_{k\in\mathbb{N}} T_k, \quad T_k := \mathrm{Span}\{X^\alpha : \alpha \in \mathbb{N}^d, |\alpha| = k\} = \bigoplus_{\alpha:|\alpha|=k} \mathrm{Span}\{X^\alpha\}$$

with the norm $\|X^\alpha\|_\alpha := 1$ on the spaces $\mathrm{Span}\{X^\alpha\}$ and the norm $\|\cdot\|_{T_k} := \sum_{\alpha:|\alpha|=k} \|\cdot\|_\alpha$ on T_k.

The binomial formula for multiindices (Appendix A) inspires to define for every $h \in \mathbb{R}^d, \alpha \in \mathbb{N}^d$,

$$(X+h)^\alpha := \sum_{\beta\in\mathbb{N}^d:0\leq\beta\leq\alpha} \binom{\alpha}{\beta} X^\beta h^{\alpha-\beta}$$

Let the structure group be

$$G := \{\Gamma_h : T \to T | h \in \mathbb{R}^d\}$$

where for every $\alpha \in \mathbb{N}^d$ and $h \in \mathbb{R}^d$ we define $\Gamma_h X^\alpha := (X+h)^\alpha$ and extend linearly to all of T. This indeed defines a group and $G \approx (\mathbb{R}^d, +)$.

Each space T_k is finite-dimensional with $\dim(T_k) = \frac{(d+k-1)!}{k!(d-1)!}$ and basis vectors corresponding to the symbols X^α with $|\alpha| = k$. We will call an element $\tau \in T_k$ an abstract polynomial of order k. The correspondence between abstract polynomials and polynomials as real-valued functions on $\mathbb{R}^d$ is clear from the following definition of the model for $\mathcal{T}_d$.

Let the model for the polynomial structure be (Π, Γ) defined for all $x, y \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$ by

$$\Pi_x X^\alpha := (\cdot - x)^\alpha \in \mathcal{C}(\mathbb{R}^d)$$

and

$$\Gamma_{xy} X^\alpha := (X + x - y)^\alpha$$

both extended linearly to all of T.

We show that these definitions satisfy the properties in Def. 3.1, 3.4. Clearly the properties 1 and 2 in Def. 3.1 are satisfied for $\mathcal{T}_d$. For property 3, we note that for any $m \in \mathbb{N}, \alpha \in \mathbb{N}^d$ such that $|\alpha| = m$,

$$\mathcal{Q}_m \Gamma_{xy} X^\alpha = \mathcal{Q}_m (X + x - y)^\alpha = \mathcal{Q}_m \sum_{\beta \in \mathbb{N}^d : 0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} X^\beta x^{\alpha - \beta} = X^\alpha$$

by the definition of $(X + x - y)^\alpha$ via the binomial formula. This shows that, indeed,

$$(\Gamma_{xy} - \mathrm{Id}) X^\alpha \in \bigoplus_{k<m} T_k$$

and we conclude $\mathcal{T}_d$ is a regularity structure.

We verify properties 1 and 2 in Def. 3.4. For any abstract polynomial X^α,

$$\Gamma_{xy} \Gamma_{yz} X^\alpha = (X + x - y + y - z)^\alpha = \Gamma_{xz} X^\alpha$$

and for any $\gamma > 0$, any $l, m \in \mathbb{N}$ with $l < m < \gamma$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = l$,

$$\begin{aligned} \|\Gamma_{xy} X^\alpha\|_m = \|(X + x - y)^\alpha\|_m &= \left\| \sum_{|\beta| \leq l} \binom{\alpha}{\beta} X^\beta (x-y)^{\alpha - \beta} \right\|_m \\ &= \left\| \sum_{|\beta| = m} \binom{\alpha}{\beta} X^\beta (x-y)^{\alpha - \beta} \right\|_m \leq \sum_{|\beta| = m} \binom{\alpha}{\beta} \|x - y\|^{l-m} \end{aligned}$$

so the analytic property of Γ is satisfied with $C_{\gamma,K} = \max\{\sum_{|\beta|=m} \binom{\alpha}{\beta} : m, l \in \mathbb{N}, m \leq l < \gamma\}$. Moreover,

$$\Pi_x \circ \Gamma_{xy} X^\alpha = \Pi_x (X + x - y)^\alpha = (\cdot - y)^\alpha = \Pi_y X^\alpha$$

and $\forall X^\alpha$ with $|\alpha| = l$, $\forall \varphi \in \mathcal{S}(B(0,1))$ with $\|\varphi\|_{\mathcal{C}^r} \leq 1$,

$$\begin{aligned} &|(\Pi_x X^\alpha)(\delta^{-d} \varphi(\delta^{-1}(\cdot - x)))| = \left| \int_{\mathbb{R}^d} (y - x)^\alpha \delta^{-d} \varphi(\delta^{-1}(y - x)) dy \right| \\ &\leq |B_\delta(0)| \delta^{-d} \sup_{x \in B_\delta(0)} |\varphi(\delta^{-1} x)| \|x\|^l \leq \delta^{l-d} |B_\delta(0)| = V_d \delta^l \end{aligned}$$

where V_d is the volume of the unit ball in $\mathbb{R}^d$. We conclude that (Π, Γ) is a model for $\mathcal{T}_d$.

Functions in $\mathcal{D}^\gamma$ with this regularity structure and model correspond precisely to the $\mathcal{C}^\gamma$ Hölder functions in the sense of the following result which also justifies the definition of the polynomial regularity structure.

Proposition 3.9 (Relationship between $\mathcal{D}^\gamma$ and $\mathcal{C}^\gamma$)**.** [6] Assuming the polynomial structure, the function $f : \mathbb{R}^d \to T$ lies in $\mathcal{D}^\gamma$ if and only if there exists a $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\varphi(x) \in \mathcal{C}^\gamma$ and for all $x \in \mathbb{R}^d$,

$$f(x) = \sum_{\alpha:|\alpha|<\gamma} \frac{1}{\alpha!}\partial^\alpha \varphi(x) X^\alpha$$

Proof. If $f \in \mathcal{D}^\gamma$ then there exist $c_\alpha : \mathbb{R}^d \to \mathbb{R}$, $|\alpha| < \gamma$ such that

$$f(x) = \sum_{\alpha:|\alpha|<\gamma} c_\alpha(x) X^\alpha$$

as an equivalence class in $\mathcal{D}^\gamma$. Setting $\varphi(x) := c_0(x)$, we claim that $\forall \alpha : |\alpha| < \gamma$, $\partial^\alpha \varphi$ exists and

$$\partial^\alpha \varphi(x) = \alpha! c_\alpha(x) \tag{23}$$

and for all K compact and $\alpha \in \mathbb{N}^d$ with $|\alpha| = \lceil \gamma \rceil - 1$,

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|\partial^\alpha \varphi(x) - \partial^\alpha \varphi(y)|}{\|x-y\|^{\gamma+1-\lceil\gamma\rceil}} < \infty \tag{24}$$

Let $j \in \mathbb{N}, 0 \le j < \gamma$ be arbitrary. Then as $f \in \mathcal{D}^\gamma$ and applying the binomial formula, for all h such that $x, x+h \in K$,

$$\sum_{\delta:|\delta|=j} \left| c_\delta(x+h) - \sum_{\alpha:|\alpha|<\gamma,\delta\le\alpha} \binom{\alpha}{\delta} h^{\alpha-\delta} c_\alpha(x) \right| = \|f(x+h) - \Gamma_h f(x)\|_j \le |\!|\!| f |\!|\!|_{\gamma,K} \|h\|^{\gamma-j}$$

Choose an arbitrary $e_i = (0,\ldots,1,\ldots,0) \in \mathbb{N}^d$ with 1 in the i-th slot and let $h = te_i$ for $t > 0$ arbitrarily small. Let $K = \bar{B}_1(x)$ the closed unit ball centered at x.

Suppose $\gamma - 1 > j$. Then for all $\delta \in \mathbb{N}^d$ with $|\delta| = j$ and all $t : |t| \le 1$,

$$\left| \frac{c_\delta(x+te_i) - c_\delta(x)}{\|te_i\|} - \sum_{\alpha:|\alpha|<\gamma,\delta<\alpha} \binom{\alpha}{\delta} \frac{(te_i)^{\alpha-\delta}}{\|te_i\|} c_\alpha(x) \right| \le |\!|\!| f |\!|\!|_{\gamma,K} \|te_i\|^{\gamma-j-1} \tag{25}$$

the expression on the left converges to 0 as $t \searrow 0$. Also,

$$\sum_{\alpha:|\alpha|<\gamma,\delta<\alpha} \binom{\alpha}{\delta} \frac{(te_i)^{\alpha-\delta}}{\|te_i\|} c_\alpha(x) \to \binom{\delta+e_i}{\delta} c_{\delta+e_i}(x) \text{ as } t \searrow 0,$$

the rest of the terms disappearing because $e_i^{e_j} = 0$ for $i \neq j$ and $\frac{(te_i)^{\alpha-\delta}}{\|te_i\|} \searrow 0$ if $|\alpha - \delta| \ge 2$. Because the first term in (25) is precisely the i-th partial derivative, we conclude that

$$\partial^{e_i} c_\delta(x) = (\delta_i + 1) \cdot c_{\delta+e_i}(x)$$

and a simple induction gives (23).

Suppose $j = \lceil \gamma \rceil - 1$. Then for all $\delta : |\delta| = j$, $t : |t| \le 1$,

$$|c_\delta(x+te_i) - c_\alpha(x)| \le |\!|\!| f |\!|\!|_{\gamma,K} |t|^{\gamma+1-\lceil\gamma\rceil}$$

which means precisely that the derivatives of order $\lceil\gamma\rceil - 1$ are $(\gamma + 1 - \lceil\gamma\rceil)$-Hölder continuous.

Conversely, if $\varphi \in \mathcal{C}^\gamma$ and

$$f(x) = \sum_{\alpha:|\alpha|<\gamma} \frac{\partial^\alpha \varphi(x)}{\alpha!} X^\alpha$$

we prove that $f \in \mathcal{D}^\gamma$ by verifying the finiteness condition in Definition 3.6 for f. Choose any compact $K \subset \mathbb{R}^n$. First,

$$\sup_{x\in K} \sup_{j<\gamma} \|f(x)\|_j = \sup_{j<\gamma} \sum_{\beta:|\beta|=j} \sup_{x\in K} \left| \frac{\partial^\beta \varphi(x)}{\beta!} \right| < \infty$$

holds as K is compact, $\partial^\beta \varphi$ is continuous, and there are only finitely many $\beta \in \mathbb{N}^d$ such that $\beta \in A$ and $|\beta|$ smaller than γ.

Second, we need to verify

$$\sup_{x,y\in K} \sup_{j<\gamma} \frac{\left\| \sum\limits_{|\alpha|<\gamma} \frac{\partial^\alpha \varphi(x) X^\alpha}{\alpha!} - \sum\limits_{|\alpha|<\gamma} \frac{\partial^\alpha \varphi(y)(X+x-y)^\alpha}{\alpha!} \right\|_j}{\|x-y\|^{\gamma-j}} < \infty$$

where we applied the property that $\Gamma_{xy} X^\alpha = (X + x - y)^\alpha$. The second term in the numerator can be written, by the binomial formula and swapping sums, as

$$\sum_{\delta:0\leq|\delta|<\gamma} \sum_{\alpha:\delta\leq\alpha,|\alpha|<\gamma} \partial^\alpha \varphi(y) \binom{\alpha}{\delta} X^\delta \frac{(x-y)^{\alpha-\delta}}{\alpha!}$$

This implies for the j-th component that (shifting the range of α)

$$\|f(x) - \Gamma_{xy} f(y)\|_j = \left| \sum_{\beta:|\beta|=j} \frac{1}{\beta!} \left(\partial^\beta \varphi(x) - \sum_{\alpha:0\leq|\alpha|<\gamma-j} \frac{(x-y)^\alpha}{\alpha!} \partial^{\alpha+\beta} \varphi(y) \right) \right|$$

Note that each summand on the right is the remainder of Taylor expansion of $\partial^\beta \varphi$ at x up to order $\lceil\gamma\rceil - 1 - j$. Moreover, the derivatives of order $\lceil\gamma\rceil - 1$ are $(\gamma + 1 - \lceil\gamma\rceil)$-Hölder continuous. By Theorem 1.8, we can conclude that for any K compact there exists $C_{\gamma,K}$ such that $\forall x, y \in K$:

$$\|f(x) - \Gamma_{xy} f(y)\|_j \leq C_{\gamma,K} \|x - y\|^{\gamma-j}$$

□

Corollary 3.10 ($\mathcal{D}^\gamma$ lift)**.** [7] For any $\gamma > 0$ and any smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ the function $\Phi : \mathbb{R}^d \to T_d$ defined as

$$\Phi(x) := \sum_{\alpha:|\alpha|<\gamma} \frac{\partial^\alpha \phi(x)}{\alpha!} X^\alpha$$

lies in $\mathcal{D}^\gamma$. We call Φ the $\mathcal{D}^\gamma$ lift of ϕ.

Proof. Since $\phi \in \mathcal{C}^\infty \subset \bigcup\limits_{\gamma\geq 0} \mathcal{C}^\gamma$, we can proceed as in the proposition. □

Finally, we mention an essential result that relates a function in $\mathcal{D}^\gamma$ to a distribution in $\mathcal{S}'$ for a general regularity structure and model. The proof can be found in [6][4][10].

Theorem 3.11 (Reconstruction Theorem)**.** [6] For any $\gamma > 0$ and a regularity structure (A, T, G) with a model (Π, Γ) there exists a unique linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{S}'(\mathbb{R}^d)$ such that $\forall K \subset \mathbb{R}^d$ compact $\exists C_{\gamma,K} > 0$ such that $\forall \varphi \in \mathcal{S}(B_1(0)) : \|\varphi\|_{\mathcal{C}^r} \leq 1, \delta \in (0, 1]$,

$$|(\mathcal{R}f - \Pi_x f)(\delta^{-d} \varphi(\delta^{-1}(\cdot - x)))| \leq C_{\gamma,K} \delta^\gamma$$

3.2 Extension of a structure by polynomials

Definition 3.12. (A partial order on regularity structures) [6] Given two regularity structures $\mathcal{T} = (A, T, G)$ and $\bar{\mathcal{T}} = (\bar{A}, \bar{T}, \bar{G})$, we say that $\mathcal{T} \subset \bar{\mathcal{T}}$ if:

1. $A \subset \bar{A}$
2. there exists an injection $i : T \hookrightarrow \bar{T}$ such that for all $\alpha \in A$: $i(T_\alpha) \subset \bar{T}_\alpha$
3. for all $g \in \bar{G}$: $g(i(T)) \subset i(T)$
4. there exists $j : \bar{G} \to G$ a surjective homomorphism such that $i \circ j(g) = g \circ i$ for all $g \in \bar{G}$.

$$\begin{array}{ccc} T & \xrightarrow{i} & \bar{T} \\ {\scriptstyle j(g)}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{i} & \bar{T} \end{array}$$

Definition 3.13. (Extension of a model) [6] Suppose that $\mathcal{T} \subset \bar{\mathcal{T}}$ and $(\Pi, \Gamma), (\bar{\Pi}, \bar{\Gamma})$ are models for $\mathcal{T}, \bar{\mathcal{T}}$ respectively. Assuming i, j as in the preceding definition, we say that $(\bar{\Pi}, \bar{\Gamma})$ extends (Π, Γ) if for all $x, y \in \mathbb{R}^d$, all $\tau \in T$:

1. $\Gamma_{xy} = j(\bar{\Gamma}_{xy})$, or equivalently $i(\Gamma_{xy}\tau) = \bar{\Gamma}_{xy}(i(\tau))$
2. $\Pi_x \tau = \bar{\Pi}_x i(\tau)$

$$\begin{array}{ccc} T & \xrightarrow{i} & \bar{T} \\ {\scriptstyle \Gamma_{xy}}\downarrow & & \downarrow{\scriptstyle \bar{\Gamma}_{xy}} \\ T & \xrightarrow{i} & \bar{T} \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{i} & \bar{T} \\ & {\scriptstyle \Pi_x}\searrow & \downarrow{\scriptstyle \bar{\Pi}_x} \\ & & \mathcal{S}'(\mathbb{R}^d) \end{array}$$

Lemma 3.14 (Product of regularity structures)**.** [6] For two regularity structures $\mathcal{T} = (A, T, G)$ and $\mathcal{T}' = (A', T', G')$ define their product as $\mathcal{T} \otimes \mathcal{T}' := (A + A', T \otimes T', G \otimes G')$. Then $\mathcal{T} \otimes \mathcal{T}'$ is again a regularity structure, $\mathcal{T} \subset \mathcal{T} \otimes \mathcal{T}'$ and $\mathcal{T}' \subset \mathcal{T} \otimes \mathcal{T}'$.

Proof. $A + A'$ is clearly locally finite, bounded below, contains 0 and $T \otimes T' = \bigoplus_{\alpha \in A, \beta \in A'} (T_\alpha \otimes T'_\beta)$ is a graded vector space with the j-th degree component $(\mathcal{T} \otimes \mathcal{T}')_j = \bigoplus_{\alpha+\beta=j} (\mathcal{T}_\alpha \otimes \mathcal{T}_\beta)$.
Suppose $g \otimes g' \in G \otimes G'$. Then for all $\tau \in T_\alpha$ with $\alpha \in A$ and $\tau' \in T'_{\alpha'}$ with $\alpha' \in A'$,

$$g(\tau) \otimes g'(\tau') - \tau \otimes \tau' = (g(\tau) - \tau) \otimes \Gamma'\tau' + \tau \otimes (g'(\tau') - \tau')$$

Since $g(\tau) - \tau \in T_\alpha^-$ and $g'(\tau') - \tau' \in T_{\alpha'}^-$, it must hold that $g(\tau) \otimes g'(\tau') - \tau \otimes \tau' \in (T \otimes T')^-_{\alpha+\alpha'}$. So $\mathcal{T} \otimes \mathcal{T}'$ is a regularity structure.

Define $i : \mathcal{T} \hookrightarrow \mathcal{T} \otimes \mathcal{T}'$ as $i(\tau) := \tau \cdot 1'$ where $1' \in T'_0$ is the identity. Then clearly for all $\alpha \in A$,

$$i(T_\alpha) = T_\alpha \otimes \{1'\} \subset (T \otimes T')_\alpha$$

and for any $g \otimes g'$,

$$g \otimes g'(T \otimes \{1'\}) = g(T) \otimes \{1'\} \subset T \otimes \{1'\} = i(T)$$

because $g(T) \subset T$.

Define $j : G \otimes G' \to G$ as $j(g \otimes g') := g$ which is clearly a surjective homomorphism. Then for all $g \otimes g'$, $\tau \in T$,

$$i \circ j(g \otimes g')(\tau) = i \circ g(\tau) = g(\tau) \otimes 1' = (g \otimes g')(\tau \otimes 1') = (g \otimes g') \circ i(\tau)$$

This shows $\mathcal{T} \subset \mathcal{T} \otimes \mathcal{T}'$ and the other inclusion follows in the same way. □

The following theorem allows us to speak of multiplication by abstract polynomials in the model space.

Theorem 3.15 (Extension with polynomials)**.** [7] Let $\mathcal{T} = (A, T, G)$ be a regularity structure with model (Π, Γ) and let $\mathcal{T}' = (A', T', G')$ denote the polynomial structure and (Π', Γ') the polynomial model. Then $\bar{\mathcal{T}} := \mathcal{T} \otimes \mathcal{T}'$ extends the polynomial regularity structure and there exists an extended model $(\bar{\Pi}, \bar{\Gamma})$ for $\bar{\mathcal{T}}$ satisfying:

1. $\forall \alpha \in \bar{A}, \tau \in T_\alpha, \beta \in \mathbb{N}^d : X^\beta \tau \in T_{\alpha+|\beta|}$
2. $\forall x, y \in \mathbb{R}^d, \tau \in \bar{T}, \beta \in \mathbb{N}^d : \bar{\Gamma}_{xy}(X^\beta \tau) = (X + x - y)^\beta \Gamma_{xy} \tau$
3. $\forall x \in \mathbb{R}^d, \tau \in \bar{T}, \beta \in \mathbb{N}^d : \bar{\Pi}_x(X^\beta \tau) = (\cdot - x)^\beta \Pi_x \tau$

where we denote $X^\beta \tau := \tau \otimes X^\beta \in \mathcal{T} \otimes \mathcal{T}'$ and $(X + h)^\beta := \sum_{\alpha: 0 \leq \alpha \leq \beta} \binom{\beta}{\alpha} h^{\beta - \alpha} X^\alpha$.

Proof. We know from Lemma 3.14 that $\bar{\mathcal{T}}$ is a regularity structure and $\mathcal{T} \subset \bar{\mathcal{T}}$. Define $\bar{\Gamma} : \bar{T} \to \bar{T}$ and $\bar{\Pi}_x : T \to \mathcal{S}'(\mathbb{R}^d)$ for all $x, y \in \mathbb{R}^d$ as

$$\bar{\Gamma}_{xy} := \Gamma_{xy} \otimes \Gamma'_{xy}, \quad \bar{\Pi}_x(\tau \otimes \tau') := \Pi_x(\tau) \cdot \Pi'_x(\tau') \quad \forall \tau\tau' \in \bar{T}$$

noting that $\bar{\Pi}_x$ is well-defined because for every $\tau \otimes \tau'$ it is a product of a Schwartz distribution $\Pi_x(\tau)$ and a polynomial $\Pi'_x(\tau')$.

It remains to show $(\bar{\Pi}, \bar{\Gamma})$ is a model which extends (Π, Γ) in the sense of Definition 3.13. The algebraic properties of $\bar{\Gamma}$ and $\bar{\Pi}$ follow from the definition of tensor product of two linear maps as follows,

$$\bar{\Gamma}_{xy} \circ \bar{\Gamma}_{yz} = (\Gamma_{xy} \otimes \Gamma'_{xy}) \circ (\Gamma_{yz} \otimes \Gamma'_{yz}) = (\Gamma_{xy} \circ \Gamma_{yz}) \otimes (\Gamma'_{xy} \circ \Gamma'_{yz}) = \Gamma_{xz} \otimes \Gamma'_{xz} = \bar{\Gamma}_{xz}$$

and

$$\bar{\Pi}_x \circ \bar{\Gamma}_{xy} = (\Pi_x \otimes \Pi'_x) \circ (\Gamma_{xy} \otimes \Gamma'_{xy}) = (\Pi_x \circ \Gamma_{xy}) \otimes (\Pi'_x \circ \Gamma'_{xy}) = \Pi_y \circ \Pi'_y = \bar{\Pi}_y$$

The definition of tensor product norm implies that for any $\gamma > 0$, K compact there exists $C_{\gamma,K}$ such that for all $\beta \in \mathbb{N}^d$ with $|\beta| = k$, $m, l \in A, n \in A', m < l, n < k, \tau \in T_l$,

$$\begin{aligned}\|\bar{\Gamma}_{xy}(\tau \otimes X^\beta)\|_{T_m \otimes T'_n} &= \|\Gamma_{xy}\tau\|_{T_m} \|\Gamma'_{xy} X^\beta\|_{T'_n} \\ &\leq C_{\gamma,K} \|\tau\|_{T_l} \|x - y\|^{k-m} \|X^\beta\|_{T'_k} \|x - y\|^{l-n} \\ &\leq C_{\gamma,K} \|\tau \otimes X^\beta\|_{T_l \otimes T'_k} \|x - y\|^{k+l-(m+n)}\end{aligned}$$

By the definition of product of a distribution and a smooth function, it holds that for all $\gamma > 0$, K compact there exists $C > 0$ such that for all $\tau \in T_l, \beta \in \mathbb{N}^d$ with $l + |\beta| < \gamma, \varphi \in \mathcal{S}(B_1(0))$ with $\|\varphi\|_{\mathcal{C}^r} \leq 1$,

$$|\bar{\Pi}(X^\beta \tau)| = |\Pi'_x(X^\beta) \cdot \Pi_x(\tau)| = |\Pi_x(\tau)(\delta^{-d}(\cdot - x)^\beta \varphi(\delta^{-1}(\cdot - x)))| \leq C_{\gamma,K} \|X^\beta \tau\|_{|\beta|+l} \delta^{-|\beta|-l},$$

concluding the proof that $(\bar{\Pi}, \bar{\Gamma})$ is a model.
Let i, j be defined as in the proof of Lemma 3.14. Then

$$j(\Gamma_{xy} \otimes \Gamma'_{xy}) = \Gamma_{xy}$$

and

$$\bar{\Pi}_x i(\tau) = \bar{\Pi}_x(\tau \otimes \{1'\}) = \Pi_x(\tau) \Pi'_x(1) = \Pi_x(\tau)$$

which proves that $(\bar{\Pi}, \bar{\Gamma})$ extends (Π, Γ).
The three properties of multiplication by polynomials are clear from the definition of $(\bar{\Pi}, \bar{\Gamma})$ in the proof. ∎

Corollary 3.16 (Multiplication estimate)**.** [7] Suppose $\tau_1 = \sum_{\alpha \in A} a_\alpha X^\alpha \in T'$ and $\tau_2 = \sum_{\alpha' \in \bar{A}} b_{\alpha'} e_{\alpha'} \in \bar{T}$ where $e_{\alpha'} \in \bar{T}_{\alpha'}, a_\alpha, b_{\alpha'} \in \mathbb{R}$. Then $\forall \beta \in \bar{A}$,

$$\|\tau_1 \tau_2\|_\beta \leq \sum_{\alpha : 0 \leq \alpha \leq \gamma} \|\tau_1\|_\alpha \|\tau_2\|_{\beta - \alpha}$$

Proof. Multiplication properties in the preceding theorem yield

$$\tau_1 \tau_2 = \sum_{\alpha \in A} \sum_{\alpha' \in \bar{A}} a_\alpha b_{\alpha'} (X^\alpha e_{\alpha'})$$

with $X^\alpha e_{\alpha'} \in \bar{T}_{\alpha + \alpha'}$. This implies

$$\|\tau_1 \tau_2\|_\beta = \left| \sum_{\alpha, \alpha' : \alpha + \alpha' = \beta} a_\alpha b_{\alpha'} \right| \leq \sum_{\alpha : 0 \leq \alpha < \gamma} |a_\alpha| |b_{\beta - \alpha}|$$

□

4 Extension theorem for modelled distributions

4.1 Statement and setup

Having presented the extension theorem for $\mathcal{C}^k$ functions in Section 2, we claim that an analogous statement holds for the modelled distributions in $\mathcal{D}^\gamma$ described in Section 3.

Theorem 4.1. (Modelled distribution extension theorem) Let $\mathcal{T} = (A, T, G)$ be a regularity structure with model (Π, Γ) and containing the polynomial regularity structure with multiplication as above. Let $\gamma \geq \min A$, let B be a closed subset of $\mathbb{R}^d$ and let $f : B \to T_\gamma^-$ be such that for all $K \subset B$ compact,

$$\sup_{x \in K} \sup_{\beta < \gamma} \|f(x)\|_\beta + \sup_{x,y \in K} \sup_{\beta < \gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x - y\|^{\gamma - \beta}} < \infty \tag{26}$$

Then there exists a $g \in \mathcal{D}^\gamma$ such that $f(x) = g(x)$ for all $x \in B$.

Remark 4.2. We will say that a function f as in Theorem 4.1 is in $\mathcal{D}^\gamma|_B$. The expression (26) defines the seminorms $|||\cdot|||_{\gamma, B\cap K}$ on the space $\mathcal{D}^\gamma|_B$.

The lift to $\mathcal{D}^\gamma$ of the functions $(\phi_\mu)_{\mu \in \mathbb{N}}$ in the partition of unity for $\mathbb{R}^d \setminus B$ will be useful.

Definition 4.3 (Lift to $\mathcal{D}^{\bar\gamma}$)**.** Let $\phi_\mu : \mathbb{R}^d \to \mathbb{R}$ be the functions defined in Section 2.2 and (A, T, G) a regularity structure containing the polynomial structure with multiplication properties as in Theorem 3.15. Define $\Phi_\mu : \mathbb{R}^d \to T_{\bar\gamma}^-$ for arbitrary $\bar\gamma > 0$ by

$$\Phi_\mu(x) = \sum_{k \in \mathbb{N}^d, |k| < \bar\gamma} \frac{\phi_\mu^{(k)}(x)}{k!} X^k$$

and $\Phi_\mu \in \mathcal{D}^{\bar\gamma}$ by Corollary 3.10.

Lemma 4.4 (A partition of unity in $\mathcal{D}^\gamma$)**.** Let (ϕ_μ) be the partition of unity for $\mathbb{R}^d \setminus B$ as in Section 2.2. Then the collection of $\mathcal{D}^\gamma$-lifted functions (Φ_μ) is still a partition of unity.

Proof. By construction, $\sum_{i \in \mathbb{N}} \phi_{\mu_i} = \sum_{\mu \in \mu(y)} \phi_\mu = 1$ and thus $\partial_\alpha \left(\sum_{\mu \in \mu(y)} \phi_\mu \right) = 0$ for all $\alpha \in \mathbb{N}^d$, $|\alpha| \geq 1$. Therefore, for all $m \geq 1$,

$$\mathcal{Q}_m \sum_{\mu \in \mu(y)} \Phi_\mu(y) = \sum_{\alpha : |\alpha| = m} \sum_{\mu \in \mu(y)} \frac{\phi_\mu^{(\alpha)}(y)}{\alpha!} = 0$$

which implies

$$\sum_{\mu \in \mu(y)} \Phi_\mu(y) = \sum_{\mu \in \mu(y)} \phi_\mu = 1 \in T_0$$

□

4.2 Proof of Extension Theorem for modelled distributions

Constants dependent only on γ and K will be merged into constants C, C', C'' whenever it is convenient.

Proof. Let $\bigcup_{k \in \mathbb{N}} S_k$ be the subdivison of $\mathbb{R}^d \setminus B$ as laid out in Section 2.2, let $(\phi_\mu)_{\mu \in \mathbb{N}}$ be the corresponding partition of unity for $\mathbb{R}^d \setminus B$ with each ϕ_μ supported in the cube I_μ centered at v^μ. Then Lemma 4.3 yields for every μ the function Φ_μ that represents ϕ_μ lifted to the $\mathcal{D}^\gamma$ space.

In analogy to Whitney's proof, set:

$$g(x) = \begin{cases} \sum_\mu \Phi_\mu(x)\Gamma_{xz^\mu} f(z^\mu) & \text{for } x \in \mathbb{R}^d \setminus B \\ f(x) & \text{for } x \in B. \end{cases}$$

We will prove that, indeed, the bound in Definition 3.6 holds for g and thus $g \in \mathcal{D}^\gamma$.
Let $K \subset \mathbb{R}^d$ be compact. Since $K \cap B$ is compact in B, the assumption that $f \in \mathcal{D}^\gamma|_B$ gives

$$\sup_{x\in K\cap B} \sup_{\beta<\gamma} \|g(x)\|_\beta + \sup_{x,y\in K\cap B} \sup_{\beta<\gamma} \frac{\|g(x) - \Gamma_{xy} g(y)\|_\beta}{\|x-y\|^{\gamma-\beta}} = \|\!|f|\!\|_{\gamma, K\cap B} < \infty$$

The following steps show this applies to the whole of K. Let $\bar{K} := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq 1\}$ denote the 1-fattening of K. This is again a compact set.

Step 1: We show that

$$\sup_{y\in K\setminus B} \sup_{\beta<\gamma} \|g(y)\|_\beta < \infty \tag{27}$$

By definition of g on $K \setminus B$,

$$\sup_{y\in K\setminus B} \sup_{\beta<\gamma} \|g(y)\|_\beta = \sup_{y\in K\setminus B} \sup_{\beta<\gamma} \left\| \sum_{\mu\in\mu(y)} \Phi_\mu(y)\Gamma_{yz^\mu} f(z^\mu) \right\|_\beta \tag{28}$$

Introduce $\Gamma_{y\pi(y)} f(\pi(y))$ which is independent of μ so that

$$\Gamma_{y\pi(y)} f(\pi(y)) = \sum_{\mu\in\mu(y)} \Phi_\mu(y)\Gamma_{y\pi(y)} f(\pi(y)) \tag{29}$$

since (Φ_μ) is a partition of unity. Then uniformly for all $y \in B \setminus K, \beta < \gamma$,

$$\begin{aligned}
&\left\| \sum_{\mu\in\mu(y)} \Phi_\mu(y)\Gamma_{yx^\mu} f(x^\mu) \right\|_\beta \leq \left\| \sum_{\mu\in\mu(y)} \Phi_\mu(y) \left(\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y))\right) \right\|_\beta + \|\Gamma_{y\pi(y)} f(\pi(y))\|_\beta \\
&\leq \sum_{\mu\in\mu(y)} \sum_{j:0\leq j<\bar{\gamma}} \|\Phi_\mu(y)\|_j \|\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y))\|_{\beta-j} + \|\Gamma_{y\pi(y)} f(\pi(y))\|_\beta \\
&\leq C_{\gamma,K} \|\!|f|\!\|_{\gamma,B\cap\bar{K}} \sum_{\mu\in\mu(y)} \sum_{j:0\leq j<\bar{\gamma}} \sum_{i:\beta-j\leq i<\gamma} \|\Phi_\mu(y)\|_j \|y - z^\mu\|^{i+j-\beta} \|z^\mu - \pi(y)\|^{\gamma-i} + \|\Gamma_{y\pi(y)} f(\pi(y))\|_\beta
\end{aligned}$$

using Lemma 3.5 and the assumption $f \in \mathcal{D}^\gamma|_B$ on the third line. By Proposition 3.10, if y lies in a cube of the subdivision S_k then for any $\mu \in \mu(y)$,

$$\|\Phi_\mu(y)\|_j \leq N 2^{k(y)j}$$

Note that $N_j \leq N$ for some N because there are finitely many j with $j < \gamma$. Since $i + j - \beta \geq 0$ and $\gamma - i \geq 0$, by the distance estimates for the subdivision in Proposition 2.16 it holds that

$$\|y - z^\mu\|^{i+j-\beta} \|z^\mu - \pi(y)\|^{\gamma-i} \leq C \cdot 2^{-k(y)(\gamma-\beta+j)} \leq C \cdot 2^{-k(y)j}$$

so the first term is bounded by $CN\|\!|f|\!\|_{\gamma,B\cap\bar{K}}$.

Second, we recall that Lemma 3.5 gives a bound on expressions of the type $\Gamma_{xy}a$. In particular, we have for the second term that there exists C' such that

$$\|\Gamma_{y\pi(y)} f(\pi(y))\|_\beta \leq C' \sum_{j:\beta\leq j<\gamma} \|f(\pi(y))\|_j \|y - \pi(y)\|^{j-\beta} \leq C'' \|\!|f|\!\|_{\gamma,B\cap\bar{K}} \tag{30}$$

uniformly for all $y \in K \setminus B, \beta < \gamma$.

It remains to verify that for all K compact there exists a $C > 0$ such that

$$\|g(x) - \Gamma_{xy} g(y)\|_\beta \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|x - y\|^{\gamma-\beta} \tag{31}$$

holds uniformly for all pairs of points x, y with $\|x-y\| \leq 1$ for the cases $\begin{cases} x, y \text{ are both in } K \setminus B \\ x \in K \cap B \text{ and } y \in K \setminus B \end{cases}$

We first prove the bound (31) holds for $x = x_0$ in $K \cap B$ and $y \in K \setminus B$. This means to show that there exists $C > 0$ such that uniformly for $y \in K \setminus B$, $x_0 \in K \cap B$ with $\|y - x_0\| \leq 1$,

$$\|g(y) - \Gamma_{yx_0} g(x_0)\|_\beta \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|y - x_0\|^{\gamma-\beta} \tag{32}$$

Step 2: If $\mathrm{dist}(y, B) \leq 1$ then the set $B \cap \bar{K}$ clearly contains $\pi(y)$ and z^μ for all $\mu \in \mu(y)$. Since $f \in \mathcal{D}^\gamma|_B$, it holds that

$$\sup_{\substack{y\in K\setminus B \\ \mathrm{dist}(y,B)\leq 1}} \sup_{\mu\in\mu(y)} \frac{\|f(z^\mu) - \Gamma_{z^\mu\pi(y)} f(\pi(y))\|_j}{\|z^\mu - \pi(y)\|^{\gamma-j}} \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} < \infty \tag{33}$$

where the constant C arises from Remark 3.7, noting that although $\pi(y), z^\mu \in B \cap \bar{K}$, we do not necessarily have $\|\pi(y) - z^\mu\| \leq 1$.

Step 3: We show that uniformly for all $y \in K \setminus B, \mu \in \mu(y)$,

$$\|\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y))\|_\beta \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} 2^{-k(y)(\gamma-\beta)} \tag{34}$$

The fact that $\Gamma_{y\pi(y)} = \Gamma_{yz^\mu} \Gamma_{z^\mu\pi(y)}$ yields

$$\begin{aligned} \|\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y))\|_\beta &= \|\Gamma_{yz^\mu} \left(f(z^\mu) - \Gamma_{z^\mu\pi(y)} f(\pi(y))\right)\|_\beta \\ &\leq C \sum_{j:\beta\leq j<\gamma} \|y - z^\mu\|^{j-\beta} \|f(z^\mu) - \Gamma_{z^\mu\pi(y)} f(\pi(y))\|_j \\ &\leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \sum_{j:\beta\leq j<\gamma} \|y - z^\mu\|^{j-\beta} \|z^\mu - \pi(y)\|^{\gamma-j} \end{aligned}$$

where we used again the argument that Γ_{yz^μ} maps to the same or lower order homogeneities and applied the estimate (33) on the third line. As $j - \beta \geq 0$ and $\gamma - j \geq 0$, we apply estimates from Proposition 2.16 to obtain (34).

Step 4: We will show that uniformly for all $y \in K \setminus B, x_0 \in K \cap B$, $\beta < \gamma$,

$$\left\| \sum_{\mu\in\mu(y)} \Phi_\mu(y) \Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y)) \right\|_\beta \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|y - x_0\|^{\gamma-\beta} \tag{35}$$

Using that (Φ_μ) is a partition of unity for $\mathbb{R}^d \setminus B$, the left side of (35) can be rewritten as

$$\left\| \sum_{\mu \in \mu(y)} \Phi_\mu(y) \left(\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y)) \right) \right\|_\beta \leq \tag{36}$$

$$\leq \sum_{\mu \in \mu(y)} \sum_{j: 0 \leq j < \bar{\gamma}} \|\Phi_\mu(y)\|_j \|\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y))\|_{\beta - j} \tag{37}$$

$$\leq C \|f\|_{\gamma, B \cap \bar{K}} \sum_{\mu \in \mu(y)} \sum_{j: 0 \leq j < \bar{\gamma}} \sum_{i: \beta - j \leq i < \gamma} \|\Phi_\mu(y)\|_j \|y - z^\mu\|^{i + j - \beta} \|z^\mu - \pi(y)\|^{\gamma - i}$$

The first inequality follows from the multiplication estimate in Corollary 3.16, the second inequality follows from Lemma 3.5. From the definition of a $\mathcal{D}^\gamma$ lift,

$$\|\Phi_\mu(y)\|_j = \frac{1}{j!} |\phi_\mu^{(j)}(y)|$$

and Proposition 2.14 gives the uniform bound on derivatives of ϕ,

$$\forall y \in \mathbb{R}^d, \mu \in \mu(y): \quad |\phi_\mu^{(j)}(y)| \leq N_j 2^{k(y)j}$$

Since $0 \leq i + j - \beta$ and $0 \leq \gamma - i$, the subdivision estimates from Proposition 2.16 imply $\exists C_1, C_2 > 0$ such that $\forall y \in K \setminus B, x_0 \in \partial B$ with $\|y - x_0\| \leq 1$, $\forall \mu \in \mu(y)$,

$$\|y - z^\mu\|^{i+j-\beta} \|z^\mu - \pi(y)\|^{\gamma - i} \leq C_2 \cdot 2^{-k(y)(\gamma - \beta + j)} \leq \frac{C_2}{C_1} \cdot 2^{-k(y)j} \|y - x_0\|^{\gamma - \beta}$$

Hence it holds for all $y \in K \setminus B, \mu \in \mu(y)$,

$$\begin{aligned} \|\Phi_\mu(y)\|_j \|y - z^\mu\|^{i+j-\beta} \|z^\mu - \pi(y)\|^{\gamma - i} &\leq N_j 2^{k(y)j} \left(\frac{C_2}{C_1} \cdot 2^{-k(y)j} \|y - x_0\|^{\gamma - \beta}\right) \\ &\leq C \|y - x_0\|^{\gamma - \beta} \end{aligned} \tag{38}$$

Inequality (38) holds for all i, j in the range of summation in the inequality (36). Therefore,

$$\left\| \sum_{\mu \in \mu(y)} \Phi_\mu(y) \left(\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{y\pi(y)} f(\pi(y)) \right) \right\|_\beta \leq C \|f\|_{\gamma, B \cap \bar{K}} \|y - x_0\|$$

uniformly for $y \in K \setminus B, x_0 \in K \cap B$ with $\|y - x_0\| \leq 1$, as required.

Step 5: The last estimate we will need is that uniformly for all $y \in K \setminus B, x_0 \in K \cap B$ with $\|y - x_0\| \leq 1$,

$$\|\Gamma_{yx_0} f(x_0) - \Gamma_{y\pi(y)} f(\pi(y))\|_\beta \leq C \|f\|_{\gamma, B \cap \bar{K}} \|y - x_0\|^{\gamma - \beta} \tag{39}$$

Repeating the argument in Step 3, we obtain

$$\|\Gamma_{yx_0} f(x_0) - \Gamma_{y\pi(y)} f(\pi(y))\|_\beta \leq C \|f\|_{\gamma, B \cap \bar{K}} \sum_{j: \beta \leq j < \gamma} \|y - x_0\|^{j - \beta} \|x_0 - \pi(y)\|^{\gamma - j}$$

and applying the estimate

$$\|x_0 - \pi(y)\| \leq \|y - x_0\| + \|y - \pi(y)\| \leq 2\|y - x_0\|$$

gives (39).

Step 6: Finally, combining estimates (35), (39) via the triangle inequality, we conclude that uniformly for all $y \in K \setminus B, x_0 \in K \cap B$ with $\|y - x_0\| \leq 1$,

$$\|g(y) - \Gamma_{yx_0} g(x_0)\|_\beta = \left\| \sum_{\mu \in \mu(y)} \Phi_\mu(y) \left(\Gamma_{yz^\mu} f(z^\mu) - \Gamma_{yx_0} f(x_0)\right) \right\|_\beta \leq C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|y - x_0\|^{\gamma-\beta}$$

Step 7: Suppose $x, y \in K \setminus B$ and without loss of generality $\mathrm{dist}(y, B) \leq \mathrm{dist}(x, B)$.We will distinguish two cases for (40) depending on the distance of x, y.

Case 1: Assume $\mu(x) \cap \mu(y) \neq \emptyset$. In this case, x, y lie either in the same cube of S_k, or in two neighbouring cubes of the same subdivision, or in two cubes of two consecutive subdivisions S_k, S_{k+1} not far apart. This means $S_{k(x)} = S_{k(y)}$ or $S_{k(x)+1} = S_{k(y)}$ and $\|x - y\| \leq C \cdot 2^{-k(y)}$. Let

$$\bar{\gamma} := \gamma - \min A \geq \gamma$$

and suppose the functions Φ_μ are lifted to the $\mathcal{D}^{\bar{\gamma}}$ space. By definition of g on $K \setminus B$, the assumed multiplication operation on $\mathcal{T}$ and the identity $\Gamma_{xy}\Gamma_{yz^\mu} = \Gamma_{xz^\mu}$,

$$\begin{aligned} g(x) - \Gamma_{xy} g(y) &= \sum_{\mu \in \mu(x)} \Phi_\mu(x) \Gamma_{xz^\mu} f(z^\mu) - \Gamma_{xy} \left(\sum_{\mu \in \mu(y)} \Phi_\mu(y) \Gamma_{yz^\mu} f(z^\mu) \right) = \\ &= \sum_{\mu \in \mu(x)} \Phi_\mu(x) \Gamma_{xz^\mu} f(z^\mu) - \sum_{\mu \in \mu(y)} (\Gamma_{xy} \Phi_\mu(y))(\Gamma_{xz^\mu} f(z^\mu)) = \\ &= \sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) \Gamma_{xz^\mu} f(z^\mu) \end{aligned} \tag{40}$$

Since $\sum_{\mu \in \mathbb{N}} \Phi_\mu = 1 \in T_0$, we have

$$\sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) = 0$$

and since $\Gamma_{xy} g(y)$ is independent of μ, also

$$\sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) \Gamma_{xy} g(y) = 0 \tag{41}$$

Using (41), rewrite (40) as

$$\sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) \Gamma_{xz^\mu} f(z^\mu) = \sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) (\Gamma_{xz^\mu} f(z^\mu) - \Gamma_{xy} g(y))$$

Then

$$\begin{aligned} &\left\| \sum_{\mu \in \mu(x) \cup \mu(y)} (\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)) (\Gamma_{xz^\mu} f(z^\mu) - \Gamma_{xy} g(y)) \right\|_\beta \leq \\ &\leq \sum_{\mu \in \mu(x) \cup \mu(y)} \sum_{j: 0 \leq j < \bar{\gamma}} \|\Phi_\mu(x) - \Gamma_{xy} \Phi_\mu(y)\|_j \|\Gamma_{xz^\mu} f(z^\mu) - \Gamma_{xy} g(y)\|_{\beta - j} \\ &\leq \sum_{\mu \in \mu(x) \cup \mu(y)} \sum_{j: 0 \leq j < \bar{\gamma}} \sum_{i: \beta - j \leq i < \gamma} (N 2^{k(x)\bar{\gamma}} \|x - y\|^{\bar{\gamma} - j}) \cdot (C_{\gamma, K} \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|x - y\|^{i + j - \beta} \|z^\mu - y\|^{\gamma - i}) \\ &= \sum_{\mu \in \mu(x) \cup \mu(y)} \sum_{j: 0 \leq j < \bar{\gamma}} \sum_{i: \beta - j \leq i < \gamma} C \|\!|f|\!\|_{\gamma, B\cap\bar{K}} \|x - y\|^{\gamma - \beta} \cdot 2^{k(x)\bar{\gamma}} \|x - y\|^{\bar{\gamma} - \gamma + i} \|z^\mu - y\|^{\gamma - i} \end{aligned}$$

where on the third line we used the identity

$$\Gamma_{xz^\mu} f(z^\mu) - \Gamma_{xy} g(y) = \Gamma_{xz^\mu}(f(z^\mu) - \Gamma_{z^\mu y} g(y))$$

and applied the estimate obtained in Step 6 for $y \in K \setminus B, z^\mu \in \partial B$.
Since $\bar{\gamma} - \gamma + i \geq 0$ and $\gamma - i \geq 0$ for all i such that $\min A \leq i < \gamma$, and

$$\|x - y\| \leq C \cdot 2^{-k(y)}, \ \|z^\mu - y\| \leq C \cdot 2^{-k(y)},$$

the first from the Case 1 assumption and the second from estimates in Proposition 2.16, we have

$$2^{k(y)\bar{\gamma}} \|x - y\|^{\bar{\gamma}-\gamma+i} \|z^\mu - y\|^{\gamma-i} \leq C$$

Because all of the sums are finite, there exists $C > 0$ such that for all $x, y \in K \setminus B$ with $\mu(x) \cap \mu(y) \neq \emptyset$,

$$\|g(x) - \Gamma_{xy} g(y)\|_\beta \leq C |||f|||_{\gamma, B\cap\bar{K}} \|x - y\|^{\gamma-\beta}$$

Case 2: Assume $\mu(x) \cap \mu(y) = \emptyset$, which implies $\|x - y\| \geq 2^{-k(y)}$. Then by the triangle inequality,

$$\begin{aligned}\|g(x) - \Gamma_{xy} g(y)\|_\beta &\leq \|g(x) - \Gamma_{x\pi(y)} f(\pi(y))\|_\beta + \|\Gamma_{x\pi(y)} f(\pi(y)) - \Gamma_{xy} g(y)\|_\beta \\ &\leq C |||f|||_{\gamma, B\cap\bar{K}} \|x - \pi(y)\|^{\gamma-\beta} + \|\Gamma_{xy}(\Gamma_{y\pi(y)} f(\pi(y)) - g(y))\|_\beta \\ &\leq C |||f|||_{\gamma, B\cap\bar{K}} \|x - \pi(y)\|^{\gamma-\beta} + C |||f|||_{\gamma, B\cap\bar{K}} \sum_{j:\beta\leq j<\gamma} \|x - y\|^{j-\beta} \|y - \pi(y)\|^{\gamma-j}\end{aligned}$$

where we used the estimates obtained in step 6 for $x \in K \setminus B, \pi(y) \in \partial B$. We know that $\exists C > 0$ such that $\|y - \pi(y)\| \leq C 2^{-k(y)}$ by the definition of the subdivision, therefore $\|y - \pi(y)\| \leq C\|x - y\|$ and also $\|x - \pi(y)\| \leq \|y - x\| + \|y - \pi(y)\| \leq C'\|y - x\|$. Hence for all $x, y \in K \setminus B$ with $\mu(x) \cap \mu(y) = \emptyset$,

$$\|g(x) - \Gamma_{xy} g(y)\|_\beta \leq C |||f|||_{\gamma, B\cap\bar{K}} \|x - y\|^{\gamma-\beta}$$

concluding both cases.

We deduce that, indeed,

$$\sup_{\substack{x,y\in K\setminus B \\ \|x-y\|\leq 1}} \sup_{\beta<\gamma} \frac{\|g(x) - \Gamma_{xy} g(y)\|_\beta}{\|x - y\|_{\gamma-\beta}} \leq C |||f|||_{\gamma, B\cap\bar{K}}$$

Step 8: We investigated all cases of pairs $x, y \in K$ for arbitrary K and $\beta < \gamma$. Since there are only finitely many $\beta \in A$ such that $\beta < \gamma$, we conclude that for all K compact,

$$\sup_{\beta<\gamma} \sup_{x\in K} \|g(x)\|_\beta + \sup_{\beta<\gamma} \sup_{x,y\in K} \frac{\|g(x) - \Gamma_{xy} f(y)\|_\beta}{\|x - y\|^{\gamma-\beta}} = |||g|||_{\gamma,K} \leq C |||f|||_{\gamma, B\cap\bar{K}} < \infty$$

□

The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.5 (Continuity of the extension operator)**.** For every B closed and every K compact there exists $C > 0$ such that for all $f \in \mathcal{D}^\gamma|_B$,

$$|||g|||_{\gamma,K} \leq C |||f|||_{\gamma, B\cap\bar{K}}$$

where g is the extension of f given in Theorem 4.1.

5 Examples and counterexamples

We start with an example that shows Whitney's Extension Theorem is a special case of Theorem 4.1 for γ non-integer.

Example 5.1. Let $\mathcal{T}_d$ be the polynomial regularity structure and let $\gamma > 0$. Then the space $\mathcal{D}^\gamma$ corresponds by Proposition 3.9 to the Hölder space $\mathcal{C}^\gamma$. In this case, the presented Modelled Distribution Extension Theorem is precisely Whitney's Extension Theorem, reasoning as follows. A given function $f \in \mathcal{C}^\gamma(B)$ can be lifted to $\hat{f} \in \mathcal{D}^\gamma(B)$ by Lemma 4.3 and extended to $\hat{g} \in \mathcal{D}^\gamma(\mathbb{R}^d)$ by Theorem 4.1. Then $g := \mathcal{Q}_0 \hat{g}$ is a $\mathcal{C}^\gamma$ extension of f because

$$f = \mathcal{Q}_0 \hat{f} = (\mathcal{Q}_0 \hat{g})|_B = g|_B$$

Furthermore, we have the estimate

$$\|g\|_{\mathcal{C}^\gamma, K} \leq C \|f\|_{\mathcal{C}^\gamma, B \cap \bar{K}}$$

We recall that the inclusion of the polynomial structure was one of the assumptions of Theorem 4.1. The following counterexample shows that an extension might not exist for structures not containing the polynomial structure.

Example 5.2. [7] Let $B = \{0, 1\} \subset \mathbb{R}$ and τ be a symbol so that $\mathrm{Span}(\tau)$ is a 1-dimensional Banach space. Define the structure $(A, T, G) = (\{-1, 0\}, \mathrm{Span}(\tau) \oplus \mathbb{R}, \{\mathrm{Id}\})$ with the model (Π, Γ) where Π is arbitrary and $\Gamma_{xy} = \mathrm{Id}$ for all $x, y \in \mathbb{R}$. Define $f : \{0, 1\} \to T$ as

$$f(0) = \tau, \qquad f(1) = 2\tau$$

f is clearly in $\mathcal{D}^\gamma$ for any $\gamma > -1$ since it is only defined on two isolated points. We will show there is no extension of f in $\mathcal{D}^\gamma$ for $\gamma > 0$. Suppose for a contradiction that there exists an extension $g : \mathbb{R} \to T$ and write

$$g(x) = g_0(x) + g_{-1}(x)\tau$$

Then

$$\|g(0)\|_0 = \|g(1)\|_0 = 0, \qquad \|g(0)\|_{-1} = 1, \|g(1)\|_{-1} = 2$$

If $g \in \mathcal{D}^\gamma$, then it must hold that for all $x, y \in [-1, 2]$,

$$\|g(x) - g(y)\|_{-1} = |g_{-1}(x) - g_{-1}(y)| \leq C \|x - y\|^{\gamma+1}$$

If $\gamma > 0$ then this implies g_{-1} is differentiable in $(-1, 2)$ and $g'_{-1} = 0$ in this interval. But then g_{-1} is constant on $[-1, 2]$ which is a contradiction as $g_{-1}(0) \neq g_{-1}(1)$.

As the next example shows, the existence of extensions depends on the properties of the set B.

Example 5.3. [7] Suppose $B = [0, 1] \subset \mathbb{R}$ and (A, T, G) as in the previous example. Let $\gamma > 0$ and $f : B \to T$ be any function in $\mathcal{D}^\gamma|_B$,

$$f(x) = f_0(x) + f_{-1}(x)\tau$$

As before, if an extension $g \in \mathcal{D}^\gamma$ exists then g_{-1} must be a constant and

$$g(x) = g_0(x) + C\tau$$

This implies that $f_{-1} = C$. An extension g_0 of f_0 exists since $f_0 \in \mathcal{C}^\gamma$ so we can apply Whitney's Extension Theorem and $g \in \mathcal{D}^\gamma$ is an extension of $f \in \mathcal{D}^\gamma|_B$. Here the regularity structure and model forced f_{-1} to be constant which can be trivially extended and the contradiction that occurred in Example 5.2 was avoided.

6 Further research directions

Analogous to the space of smooth functions $\mathcal{C}^\infty$, we can define the space $\mathcal{D}^\infty$.

Definition 6.1. [7] We say that $f \in \mathcal{D}^\infty$ if $\bar{\mathcal{Q}}_\gamma f \in \mathcal{D}^\gamma$ for all $\gamma > \min A$. Similarly, we define $f \in \mathcal{D}^\infty|_B$ for a closed set B.

Further extrapolating Whitney's original work, one should also be able to prove the following.

Conjecture 6.2 (Modelled distribution extension for $\mathcal{D}^\infty$)**.** Suppose $f \in \mathcal{D}^\infty|_B$. Then there exists a $g \in \mathcal{D}^\infty(\mathbb{R}^d)$ such that $f(x) = g(x)$ for all $x \in B$.

We believe this can be proved by an approach similar to the proof of Theorem 4.1 and using the technique expounded in the rest of the publication [11].

Finally, the examples in section 5 illustrate that a $\mathcal{D}^\gamma$ may not have an extension without including the polynomial structure. Moreover, the examples suggest that connectedness of the set B plays a role for the existence of an extension. One could further investigate the precise conditions on B under which there exists or does not exist an extension.

A Multiindices

Definition A.1 (Multiindex notation)**.** [8] [11] For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ define $\alpha! := \alpha_1! \dots \alpha_d!$ and the length of α

$$|\alpha| := \alpha_1 + \dots \alpha_d$$

Define the partial derivative

$$\partial^\alpha := \partial^{\alpha_1} \dots \partial^{\alpha_d}$$

as an operator on the space of $|\alpha|$-times continuously differentiable functions.

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ define

$$x^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$$

Definition A.2 (Binomial coefficients for multiindices)**.** [8] [11] For $n, k \in \mathbb{N}^d$ multiindices such that $k \leq n$ in the componentwise sense, define

$$\binom{n}{k} := \prod_{i=1}^{d} \binom{n_i}{k_i}$$

Proposition A.3 (Binomial theorem for multiindices)**.** Let $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}^d$ a multiindex. Then

$$(x+y)^n = \sum_{k \in \mathbb{N}^d : 0 \leq k \leq n} \binom{n}{k} x^k y^{n-k}$$

Proof. By the simple binomial theorem for real numbers,

$$\begin{aligned}(x+y)^n &= \prod_{i=1}^{d} \left(\sum_{k_i=0}^{n_i} \binom{n_i}{k_i} x_i^{k_i} y_i^{n_i-k_i} \right) = \sum_{k_1=0}^{n_1} \dots \sum_{k_d=0}^{n_d} \left(\prod_{i=1}^{d} \binom{n_i}{k_i} x_i^{k_i} y_i^{n_i-k_i} \right) \\ &= \sum_{k_1=0}^{n_1} \dots \sum_{k_d=0}^{n_d} \left(\binom{n}{k} x^k y^{n-k} \right) = \sum_{k \in \mathbb{N}^d : 0 \leq k \leq n} \binom{n}{k} x^k y^{n-k}\end{aligned}$$

where we denoted the multiindices $k = (k_1, \dots, k_d), n = (n_1, \dots, n_d)$. □

B Tensor product

Definition B.1 (Tensor product of vector spaces)**.** [9] Let A, B be vector spaces over $\mathbb{R}$. Let $F(A \times B)$ be the free abelian group on their Cartesian product. Define the relation $\sim$ as $\forall a, a' \in A, b, b' \in B, r \in \mathbb{R}$,

1. $(a + a', b) \sim (a, b) + (a', b)$
2. $(a, b + b') \sim (a, b) + (a, b')$
3. $(ra, b) \sim (a, rb)$

Then $A \otimes B := F(A \times B)/\sim$ defines the tensor product of A and B.

Definition B.2 (Tensor product of linear maps)**.** [9] Suppose A, B, X, Y are vector spaces and $F : A \to X$, $G : B \to Y$ are linear. Define $F \otimes G : A \otimes B \to X \otimes Y$ as

$$(F \otimes G)(a \otimes b) := F(a) \otimes G(b)$$

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