Measurable functions - foundation

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1 Some topological facts

Definition 1.1 (Basis and subbasis)**.** Let (X, τ) be a topological space. A family of sets $\mathcal{B} \subset \tau$ is called a basis if every set $A \in \tau$ can be written as an arbitrary union of sets in $\mathcal{B}$.
A family of sets $\mathcal{B}' \subset \tau$ is called a subbasis if every set $A \in \tau$ can be written as an arbitrary union of finite intersections of sets in $\mathcal{B}'$.

Definition 1.2 (Second-countable)**.** A topological space is called second-countable if there exists a countable basis for its topology.

Proposition 1.3. The Euclidean topology of $\mathbb{R}$ is generated by the countable basis $\mathcal{B} := \{(a, b) : a, b \in \mathbb{Q}\}$.

Proof. Let U be an open set in $\mathbb{R}$. Take arbitrary $x \in U$. As U is open, there exist $u_x, v_x \in \mathbb{R}$ such that $x \in (u_x, v_x) \subset U$. Between any two real numbers there is a rational number, hence there are $a_x, b_x \in \mathbb{Q}$ s.t. $x \in (a_x, b_x) \subset (u_x, v_x) \subset U$. This implies that

$$U = \bigcup_{x \in U} (a_x, b_x)$$

Since a_x, b_x are rational, this in fact reduces to a countable union. □

Corollary 1.4. The family $\mathcal{B}' := \{(a, \infty) : a \in \mathbb{Q}\} \cup \{(-\infty, b] : b \in \mathbb{Q}\}$ is a countable subbasis for $\mathbb{R}$.

Proof. We can write $(a, b] = (a, \infty) \cap (-\infty, b]$ for any $a, b \in \mathbb{Q}$. Note that these are half-open intervals. Then $(a, b) = \bigcup_{k \in \mathbb{N}} (a, b - \frac{1}{k}]$. So any element in $\mathcal{B}$ can be written as a countable union of finite intersections of elements in $\mathcal{B}'$. But $\mathcal{B}$ is a countable basis as shown above, hence $\mathcal{B}'$ is a countable subbasis. □

Remark 1.5. This can easily be generalized to $\mathbb{R}^d$; the countable basis would be the open cubes with vertices at points with rational coordinates.

2 Measurable functions

Lemma 2.1. Let $(X, \mathcal{M})$ be a measurable space, Y a set and $f : X \to Y$ a mapping. Then

$$\mathcal{F} := \{A \subset Y : f^{-1}(A) \in \mathcal{M}\}$$

is a σ-algebra on Y.

Proof. $f^{-1}(\emptyset) = \emptyset \in \mathcal{M}$ implies $\emptyset \in \mathcal{F}$.
Suppose $A \subset Y$ is such that $f^{-1}(A) \in \mathcal{M}$. Then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in \mathcal{M}$ implies $Y \setminus A \in \mathcal{F}$.
Suppose $(A_i)_{i \in \mathbb{N}}, A_i \subset Y$ are such that $f^{-1}(A_i) \in \mathcal{M}$. Then $f^{-1}(\bigcup A_i) = \bigcup f^{-1}(A_i) \in \mathcal{M}$ implies $\bigcup A_i \in \mathcal{F}$.
Hence $\mathcal{F}$ is a σ-algebra. □

Remark 2.2. The following condition is particularly suitable for verifying the measurability of $\sup_n f_n$.

Lemma 2.3 (A sufficient condition)**.** A function $f : X \to \mathbb{R}$ is $\mathcal{M}/\mathcal{B}(\mathbb{R})$-measurable if

$$f^{-1}((a,\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{Q} \tag{1}$$

Proof. If (1) holds then we also have $f^{-1}((-\infty, a]) = X \setminus f^{-1}((a,\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{Q}$. This implies

$$f^{-1}((a,b]) = f^{-1}((a,\infty)) \cap f^{-1}((-\infty, b]) \in \mathcal{M} \quad \forall a, b \in \mathbb{Q}$$

and hence

$$f^{-1}((a,b)) = \bigcup_{k \in \mathbb{N}} f^{-1}((a, b - \frac{1}{k}]) \in \mathcal{M} \quad \forall a, b \in \mathbb{Q}$$

Now, since $\mathcal{B} = \{(a,b) : a, b \in \mathbb{Q}\}$ is a countable basis, we can write for any open $U \subset \mathbb{R}$,

$$U = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$$

for some $a_i, b_i \in \mathbb{Q}$. Then

$$f^{-1}(U) = \bigcup_{i \in \mathbb{N}} f^{-1}((a_i, b_i)) \in \mathcal{M}$$

by the above. This implies that the σ-algebra $\mathcal{F}$, as defined in the previous Lemma, contains all open sets in $\mathbb{R}$, i.e.

$$\{U : U \text{ open in } \mathbb{R}\} \subset \mathcal{F}$$

But $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that contains all open sets, hence

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$$

which means precisely that f is measurable. □

Remark 2.4. The fact that $\mathcal{B}'$ is a countable subbasis for $\mathbb{R}$ was essential for the proof - we were working with countable unions and intersections all along.

Proposition 2.5. Let $(X, \mathcal{M})$ be a measurable space and $(f_n)_{n \geq 1}$ a sequence of measurable maps

$$f_n : (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

Then $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are all $\mathcal{M}/\mathcal{B}(\mathbb{R})$-measurable.

Proof. Let's prove measurability of $\sup_n f_n$, the rest will follow. We verify the condition (1):

$$\begin{aligned}(\sup_n f_n)^{-1}((a,\infty)) &= \{x \in X : \exists n \in \mathbb{N} \text{ s.t. } f_n(x) > a\} \\ &= \bigcup_{n \in \mathbb{N}} \{x \in X : f_n(x) > a\} \\ &= \bigcup_{n \in \mathbb{N}} f_n^{-1}((a,\infty)) \in \mathcal{M}\end{aligned}$$

Hence, by Lemma 2.3, $\sup_n f_n$ is measurable.
The fact that $\inf_n f_n$ is measurable follows from $\inf_n f_n = -\sup_n(-f_n)$ and similarly

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k, \quad \liminf_n f_n = \sup_n \inf_{k \geq n} f_k$$

are measurable. □