Local H^1 solutions of a McKean-Vlasov equation with positive feedback

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Abstract

We summarize a recent development in the analysis of a self-exciting McKean-Vlasov model of a contagious system. It was shown by (Hambly, Ledger, Søjmark, 2018) that there exists a unique, weakly differentiable local solution of this problem if given a well-behaved initial law. This was undertaken as a step towards proving the more general conjecture for the existence of a global, cadlag, piecewise weakly differentiable solution that is unique in the class of so-called physical solutions. The original work was further expanded on by two more publications by (Ledger, Søjmark, 2018).

1 Problem setting

The article [2] (Hambly et al.) deals with the McKean-Vlasov problem

$$\begin{aligned} X_t &= X_0 + W_t - \alpha L_t \\ L_t &= \mathbb{P}(\tau \leq t) \end{aligned} \tag{1}$$

where $\tau := \inf\{t \geq 0 : X_t \leq 0\}$ is a hitting time and W is a standard 1-dimensional Wiener process. A solution to this problem is a (deterministic) process L satisfying (1). Clearly L, W determine the process X pathwise.

This problem is a reformulation of the problem discussed in [1] and is a large population limit of a particle system model. One is interested in solutions up to the hitting time τ and hence we consider the law of the stopped process $\nu_t(\cdot) := \mathbb{P}(X_t \mathbb{1}_{t<\tau} \in \cdot)$ in the following. It can be shown that the law v_t admits a density V_t on $\mathbb{R}$ for $t > 0$.

A simple argument in [2] demonstrates that if $\alpha > 2\int_0^\infty x\nu_0(dx)$, meaning that the contagion effect is too large, then there are no global continuous solutions to problem (1). Hence, if we look for global solutions, it is natural to consider the space of cadlag functions instead. Here the issue of choosing a suitable jump process $\Delta L_t := L_t - L_{t-}$. The jump process is partially determined by the restriction of the process X_t to $\mathbb{R}_+$ since at the time of a jump towards 0 (which is always of size $-\alpha\Delta L_t$), the mass $\nu_{t-}(0, \alpha\Delta L_t)$ has to be deleted. The new initial law for the continuation of the process beyond the jump time t is then defined to be the law of the process right before the jump but shifted by the jump size, restricted to $\mathbb{R}_+$ and normalized to 1. Explicitly,

$$\nu_t(\cdot) := \frac{1}{1 - \nu_{t-}(0, \alpha\Delta L_t)} \nu_{t-}((\cdot - \alpha\Delta L_t) \cap \mathbb{R}_+)$$

Then the simple fact that $\Delta X_t = -\alpha\Delta L_t$ and $\Delta L_t = \mathbb{P}(\tau = t)$ implies

$$\nu_{t-}(0, \alpha\Delta L_t) = \mathbb{P}(0 < X_{t-} < \alpha\Delta L_t) = \mathbb{P}(X_{t-} > 0 \wedge X_t < 0) = \mathbb{P}(\tau = t) = \Delta L_t$$

The equation

$$\nu_{t-}(0, \alpha\Delta L_t) = \Delta L_t$$

thus gives a necessary condition for the jump process of a global solution. However, there is not a unique jump process ΔL_t satisfying this condition. Hence, the notion of a physical solution is introduced.

Definition 1.1 (Physical solution). [1] The cadlag function $L : \mathbb{R}_+ \to [0,1]$ is a physical solution to (1) if it satisfies

$$\Delta L_t = \inf\{x \geq 0 : \nu_{t-}(0, \alpha x) < x\}$$

It is conjectured that there exists a *global, unique* physical solution L_t that is weakly differentiable between jumps.

As a step towards this goal, the work [2] addresses local existence and uniqueness of solutions $L \in H^1$ to the problem (1) up to the explosion time $\sup\{t \geq 0 : \|L_t\|_{H^1} < \infty\}$ with the assumption that the density V_0 of the law of the initial variable X_0 vanishes locally at 0 with the power law. Moreover, it is shown that this local solution is unique in the class of cadlag solutions. That is, any local cadlag solution is in fact the unique weakly differentiable solution.

A major obstacle to proving the global solution conjecture is that after a jump time t the density V_t of ν_t no longer vanishes near the boundary $x = 0$. However, it has not yet been proven that such local solutions exist in H^1 and are unique in the class of cadlag solutions.

2 Local H^1 solutions

The strategy that was used in [2] to show the existence and uniqueness of local H^1 solutions with the assumption of vanishing initial laws consisted in applying the Banach Fixed Point Theorem with the following map:

Definition 2.1 (Solution map/fixed point map Γ). [2] Define the map

$$\begin{aligned} \Gamma : L^\infty(0, t_0) &\to L^\infty(0, t_0) \\ \Gamma[l]_t &:= \mathbb{P}(\tau^l \leq t) \end{aligned}$$

where the stopping time τ^l and the process (X^l_t) solve the equation

$$\begin{aligned} X^l_t &= X_0 + W_t - \alpha l_t \\ \tau^l &= \inf\{t \geq 0 : X^l_t \leq 0\} \end{aligned}$$

for a given function $l_t \in L^\infty(0, t_0)$.

Remark 2.2. If a fixed point $L \in L^\infty(0, t_0)$ with $L_t = \Gamma[L]_t$ exists, then it is clearly a solution to problem (1).

To obtain a regularity result from this approach, the map Γ is restricted to a subset of candidate solutions in H^1:

Definition 2.3 (The spaces of candidate solutions). [2] For any $\gamma \in (0, \frac{1}{2}), A > 0, t_0 > 0$, define

$$\mathcal{S}(\gamma, A, t_0) := \{l \in H^1(0, t_0) : l'_t \leq At^{-\gamma} \text{ for almost every } t \in [0, t_0]\}$$

Remark 2.4. The space $H^1(0, t_0)$ coincides with the space of absolutely continuous functions on $[0, t_0]$, hence $\mathcal{S}(\gamma, A, t_0) \subset L^\infty(0, t_0)$.

The stability of Γ on these subsets then hinges on the power law decay of the initial distribution near the boundary $x = 0$:

Assumption 2.5 (Power law decay of initial density near the origin). [2] It is assumed that $\exists C, D, x_*, \beta > 0$ such that

$$V_0(x) \leq Cx^\beta \mathbb{1}_{\{x \leq x_*\}} + D\mathbb{1}_{\{x > x_*\}} \quad \forall x \in \mathbb{R}$$

Remark 2.6. Note that $\Gamma[l]_t$ is simply the cumulative distribution function of τ^l. Hence, one can integrate with respect to $d\Gamma[l]_t$, the distribution of τ^l on $[0,\infty)$. Moreover, if $\Gamma[l]_t \in H^1(0,t_0)$ for some $t_0 > 0$ then

$$d\Gamma[l]_t = \Gamma[l]'_t dt$$

is a sub-probability measure on $[0,t_0)$.

It is shown that the map Γ is contractive on the space of continuous functions with respect to the L^∞ norm and stable on certain subsets $\mathcal{S} \subset H^1(0,t_0)$.

With this setup, the main result of [2] can then be summarized as follows:

Proposition 2.7 (Contractivity and stability of the fixed point map)**.** [2] There exist $K > 0$ such that for all $\epsilon > 0$ there is a $t_0 > 0$ such that the map

$$\Gamma : \mathcal{S}\left(\frac{1-\beta}{2}, K+\epsilon, t_0\right) \to \mathcal{S}\left(\frac{1-\beta}{2}, K+\epsilon, t_0\right)$$

is a well-defined contraction with respect to the L^∞ norm. Moreover, the solution to the fixed point problem $L = \Gamma[L]$ is minimal in the class of cadlag solutions, that is,

$$\bar{L}_t \geq L_t, \quad t \in [0,t_0]$$

for any cadlag solution $\bar{L}_t$ to the problem (1).

Corollary 2.8. If $l_t^{(0)} := 0$ and $l_t^{(n+1)} := \Gamma[l^{(n)}]_t$ then by the Banach Fixed Point Theorem there exists a unique $L_t \in L^\infty(0,t_0)$ such that $l^{(n)} \xrightarrow[n\to\infty]{} L$ in $L^\infty(0,t_0)$ and $\Gamma[L]_t = L_t$ for almost every $t \in [0,t_0)$. Hence L_t solves the original problem. Moreover, the fixed point L_t lies in $\mathcal{S}(\frac{1-\beta}{2}, L+\epsilon, t_0)$.

Remark 2.9. The subsets $\mathcal{S} \subset H^1$ may not be complete, so it needed to be shown *additionally* that the sequence $\{l^{(n)}\}_{n\in\mathbb{N}}$ converges to a point which lies in $\mathcal{S}$.

More specifically, the proof of Proposition 2.7 consists of two parts:

- proving that Γ is a contraction in the $L^\infty(0,t_0)$ norm
- proving that Γ is stable for the spaces $\mathcal{S}$ specified in the proposition.

The first item is precisely the following result:

Lemma 2.10 (Γ is contractive)**.** [2] For all $\gamma \in (0,\frac{1}{2}), A > 0$ there exists $t_0 > 0$ such that for all $l, \bar{l} \in \mathcal{S}(\gamma, A, t_0)$:

$$\|\Gamma[l] - \Gamma[\bar{l}]\|_{L^\infty(0,t_0)} \leq \frac{1}{2}\|l - \bar{l}\|_{L^\infty(0,t_0)}$$

This result is established by using the Reflection principle applied to the driving Brownian motion.

Second, the stability of this map is shown by the method of difference quotients (Evans [5]), as per the following result:

Lemma 2.11 (A criterion for weak differentiability)**.** [5] Suppose that $u \in H^1(0,t_0)$ and

$$\limsup_{\delta\searrow 0} \int_0^{t_0} \left|\frac{u(x+\delta) - u(x)}{\delta}\right| dx < \infty$$

Then $u \in H^1(0,t_0)$.

Proving stability of Γ then consists of multiple steps in estimating the expression

$$\left|\frac{\Gamma[l]_{t+\delta} - \Gamma[l]_t}{\delta}\right|$$

in showing that it is finite, and moreover that

$$\Gamma[l]'_t \leq (K+\epsilon)t^{-\frac{1-\beta}{2}}$$

so that, indeed, $\Gamma[l] \in \mathcal{S}(\frac{1-\beta}{2}, K+\epsilon, t_0)$. The assumption of power law decay of the initial density is crucial in showing stability.

3 H^1 solutions up to blow-up time and their uniqueness

The work [2] shows that not only can we obtain unique local solutions, but in fact unique solutions up to the blow-up time

$$t_e := \sup\{t > 0 : \|L\|_{H^1(0,t)<\infty}\}$$

This is due to the following result:

Lemma 3.1 (Power law decay property)**.** If $L_t \in H^1(0, t_0)$ is a solution to the problem (1) with initial density V_0 satisfying the assumption 2.5 with coefficients $\beta, C, D, x_* > 0$ then the density V_{t_0-} also satisfies the assumption 2.5 with the same coefficient β but with different coefficients $C', D', x'_* > 0$.

This means that, just as before, there exists a solution $L't \in H^(t_0, t_1)$ to the problem (1) started from t_0 with initial law ν_{t_0-}. These two solutions can be glued together to yield a solution $L_t \in H^(0, t_1)$. Moreover, one can show that the inverse power law decay of the weak derivative is preserved, which is important for the continuation of this argument for a sequence of times $t_n \to t_e$ [2]:

Corollary 3.2 (Closedness of the extensions in the space of candidate solutions)**.** If $L \in \mathcal{S}(\frac{1-\beta}{2}, K_0, t_0) \subset H^1$ then there exist $K_1 > 0$ such that $L \in \mathcal{S}(\frac{1-\beta}{2}, K_1, t_1)$ for the extended function L.

Combining these two results one can indeed obtain the extended solution $L \in H^1(0, t_e)$. To show uniqueness of such solution in the space of cadlag processes, the authors in [2] introduce the technique of "trapping" a hypothetical cadlag solution $\bar{L}^\epsilon$ between the previously found solution $L \in H^1$ and another process that converges to it uniformly. For this purpose, the ϵ-deleted solutions are introduced in [2].

Definition 3.3 (ϵ-deleted solutions)**.** L^ϵ is defined to be the solution to the problem:

$$\begin{aligned} X_t^\epsilon &= X_0 \mathbb{1}_{X_0 \geq \epsilon} - \frac{\epsilon}{4} + B_t - \alpha L_t^\epsilon \\ \tau^\epsilon &= \inf\{t \geq 0 : X_t^\epsilon \leq 0\} \\ L_t^\epsilon &= \nu_0(0, \epsilon) + \int_\epsilon^\infty \mathbb{P}(\tau^\epsilon \leq t)\nu_0(dx) \end{aligned}$$

It is shown that solutions $L_t^\epsilon \in \mathcal{S}(\frac{1-\beta}{2}, K, t_0)$ exist for some $K, t_0, \epsilon_0 > 0$ and all $\epsilon < \epsilon_0$ and moreover that if $L_t, L_t^\epsilon \in \mathcal{S}(\gamma, K, t_0)$ for some $\gamma, K, t_0 > 0$ then

$$\|L - L^\epsilon\|_{L^\infty(0,t_1)} \to 0 \text{ as } \epsilon \to 0$$

for some $t_1 \leq t_0$.

Finally, one can show that for any cadlag solution $\bar{L}$ up to time t_e there exist $K, t_0, \epsilon_0 > 0$ such that

$$L_t \leq \bar{L}_t < L_t^\epsilon, \quad \forall t < t_0, \epsilon < \epsilon_0$$

for the previously established solution $L \in H^1(0, t_0)$ to problem (1). Then a hypothetical cadlag solution $\bar{L}$ is trapped between L_t and L_t^ϵ on a subset $[0, t_1]$ for a $t_1 \leq t_0$. Hence, taking the limit $\epsilon \to 0$, the authors conclude that $L_t = \bar{L}_t$ on $[0, t_1]$. Using a similar bootstrapping argument as before, one can extend this uniqueness statement up to the explosion time t_e.

4 Later development

The first of the two, previously mentioned extensions of the model appends a common noise to the process, such that the system under observation becomes

$$X_t = X_0 + \sqrt{1-\rho^2} B_t + \rho B_t^0 - \alpha L_t \tag{2}$$

$$\tau = \inf\{t \geq 0 : X_t \leq 0\} \tag{3}$$

$$L_t = \mathbb{P}(\tau \leq t | B^0) \tag{4}$$

where $\rho \in [0, 1)$ and the common noise B^0 takes the form of a Brownian motion, independent of B_t. Clearly, for $\rho = 1$, this is precisely the same system (1) as before.

In [3], it is shown that the common noise has the ability to provoke or prevent blow-ups. The main result of the paper shows that initial conditions can be found that do not blow up for $\rho = 0$ but have a non-zero probability of blow-up for $\rho > 0$; more precisely, the noise provokes the blowup by quickly transporting the initial mass to the boundary. In addition, solutions to the relaxed version of (2) exist and are indeed the limit points of the associated finite particle system with positive feedback by which the applications above are motivated, although uniqueness is left undiscovered.

An additional variant allows for a continuous function $f : [0, 1] \to \mathbb{R}$ of the cadlag process L_t to be taken, giving the system:

$$\begin{aligned} X_t &= X_0 + Z_t - \alpha f(L_t) \\ \tau &= \inf\{t \geq 0 : X_t \leq 0\} \\ L_t &= \mathbb{P}(\tau \leq t) \end{aligned}$$

When f is non-negative and increasing, the closer the particle is to the boundary the higher the probability of absorption as it is pushed even closer. If these conditions on f hold true as well as insisting $f(x) = x$ and $B_t = Z_t$, this is, again, precisely the same system as before, namely, contagious with positive feedback.

In [4], several results are shown that make headway towards the conjecture stated previously. The first is global uniqueness of solutions to the second extended model above under the weak feedback regime (a smallness condition for α) and, as a result, under this regime, global well-posedness of the model with common noise (2). The second is local uniqueness of the extended model for $\alpha > 0$, $f(x) = x$ and $B_t = Z_t$ under assumptions on the initial density near zero and, as a result, under these conditions, local uniqueness of the initial problem (1) after a blow-up.

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